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METHOD OF INITIAL FUNCTIONS FOR TWO DIMENSIONAL BOUNDARY PROBLE--ETC(U)
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METHOD OF INITIAL FUNCTIONS FOR TWO DIMENSIONAL BOUNDARY PROBLEMS OF THE THEORY OF ELASTICITY

by

V. A. Agarev

FOREIGN TECHNOLOGY DIVISION

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METHOD OF INITIAL FUNCTIONS FOR TWO DIMENSIONAL BOUNDARY PROBLEMS OF THE THEORY OF ELASTICITY

By: V. A. Agarev

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U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

*ye initially, after vowels, and after ъ, ь; e elsewhere. When written as ё in Russian, transliterate as yë or ë. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

GREEK ALPHABET

Alpha	Α α *α*	Nu	Ν ν
Beta	Β β	Xi	Ξ ξ
Gamma	Γ γ	Omicron	Ο ο
Delta	Δ δ	Pi	Π π
Epsilon	Ε ε *ϵ*	Rho	Ρ ρ *ρ*
Zeta	Ζ ζ	Sigma	Σ σ *ς*
Eta	Η η	Tau	Τ τ
Theta	Θ θ *ϑ*	Upsilon	Υ υ
Iota	Ι ι	Phi	Φ *φ* ϕ
Kappa	Κ ϰ κ *κ*	Chi	Χ χ
Lambda	Λ λ	Psi	Ψ ψ
Mu	Μ μ	Omega	Ω ω

RUSSIAN AND ENGLISH TRIGONOMETRIC FUNCTIONS

Russian	English
sin	sin
cos	cos
tg	tan
ctg	cot
sec	sec
cosec	csc
sh	sinh
ch	cosh
th	tanh
cth	coth
sch	sech
csch	csch
arc sin	$\sin^{-1}$
arc cos	$\cos^{-1}$
arc tg	$\tan^{-1}$
arc ctg	$\cot^{-1}$
arc sec	$\sec^{-1}$
arc cosec	$\csc^{-1}$
arc sh	$\sinh^{-1}$
arc ch	$\cosh^{-1}$
arc th	$\tanh^{-1}$
arc cth	$\coth^{-1}$
arc sch	sech^{-1}
arc csch	csch^{-1}
rot	curl
lg	log

GRAPHICS DISCLAIMER

All figures, graphics, tables, equations, etc. merged into this translation were extracted from the best quality copy available.

Page 2.

In the book is generalized the material on the development of the method of the initial functions and is presented the for the first time developed by the author general theory of the operators in connection with the methods of the initial functions, that makes it possible to obtain the fundamentally new for this method forms of the general solutions of some procedurally important classes of the tasks of the applied theory of elasticity and strength of materials. The developed theory of the operators is illustrated by the rough estimates, in which effectively realize themselves proposed the operators.

The book is intended to scientific-technical workers, and also to the instructors, the graduate students and the students of the old courses, which specialize in elasticity theory.

Chief editor, member of AS UkSSR Pisarenko G. S.

Page 3.

Preface

During the solution of many important problems of mechanics in accordance with the contemporary demands of science the technicians are extremely effective the approximation methods, which provide for the use of computers. The effectiveness of the practical use of those or other methods in many respects depends on the method of obtaining of solutions, simplicity of the required in this case lining/calculations, etc.

One of the promising approximation methods of the solution to the boundary-value problems of elasticity theory is the method of the initial functions, which finds recently increasing application/use. However, the mathematical formalism of this method is until recently developed extremely weakly.

In the monograph, which is the dissertation work of prematurely passing away Victor Andreevich AGAREVA, is based wide use in the method of the initial functions of symbolic operations, and also is set forth the for the first time developed by the author general theory of the operators (in connection with the methods of the initial functions), which makes it possible to obtain the fundamentally new for this method forms of the general solutions of some classes of the problems of the applied theory of elasticity and

strength of materials, which have large practical application/appendix in engineering.

In the proposed theory of the operators the author substantiated the algebraic, differential and integral actions above the regular, singular and mixed operators, is demonstrated the equivalency of the representation of the operators in the closed form and in the form of series is demonstrated the legitimacy of the action above operational series, are establish/installed the specific rules for the actions with the operators, connected with the noninterchangeability of the mixed operators and so forth, and also are proposed the methods of the realization of the operators and so forth, and also are proposed the methods of the realization of the operators.

The developed theory of the operators allowed the author to obtain some new identities for Bessel functions, to obtain the new methods of the determination of the particular solutions to nonhomogeneous differential equations in partial derivatives, to reveal/detect the equivalency of operational equations and functional equations of the type integrodifferential, differential-difference and so forth.

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The author synthesized the overall diagram of the application/use of a method of the initial functions to the solution to two-dimensional boundary-value problems with the wide use of the operators. In this case were constructed general solutions for the series of problems of mechanics, including of two-dimensional problem, bending of plates, etc.

The set-forth general theory of the operators is illustrated by practical examples with finishing/bringing some of them to number.

V. A. Agareva's present monograph is the original investigations of the author in the field of applied mathematics the mechanics, who have large theoretical and practical value.

It is possible not to doubt that the published work, created gifted scientist, which was V. A. Agarev, it will be important stage in the development not only of the applied theory of elasticity and strength of materials, but also the mechanics in the broad sense of this word.

Page 5.

From the author.

Along with the increasing interest in setting and solution of nonlinear problems is given at present still considerable attention to the problems of linear problems. In accordance with the contemporary demands of science technicians in this range it is possible to indicate the following, in our opinion, main directions.

1. The formulation of the fundamentally new problems, which lead to such equations and the boundary conditions, which were not examined in classical mathematical physics.

2. Obtaining the so-called exact solutions for those problems, which have long been placed, but solution by their known at present methods is extremely difficult (due to the fundamental complexity of boundary conditions).

3. Development of the approximate (mainly numerical) methods and their adjustment to the wide use of contemporary computer technology.

4. Obtaining new forms for the known precise or approximate solutions. This is connected either with simpler - according to idea or with lining/calculations - by the method of obtaining solution or with the measure of the effectiveness of solution (in the sense of obtaining the numerical result, what, in the final analysis, frequently is basic for a practice).

The method of the initial functions (in that form, how it is now developed) it can be referred to the last/latter two directions.

In proposed work carried out the generalization of the available in known to us literature material on the application/use of a method of the initial functions; is undertaken the attempt (apparently, for the first time) to base the widely utilized in this method symbolic operations, what, in the opinion of the author, was the basic goal of his work; were given the fundamentally new for this method forms of the general solutions of some classes of the problems of the applied theory of elasticity and strength of materials, and, finally, the quality of illustration solved several concrete/specific/actual engineer missions.

The significant part of the obtained in work results is illuminated in articles [1-3, 10, 41 and 42]. Furthermore, the large part of the material of 3-4 chapters was reported by the author on seminar on mechanics with OTN [ОТН - Department of Technical Sciences] of AS UkSSR on 14 February 1959; a content 1 and 2 chapters - on the same seminar on 9 March 1962.

In conclusion I consider my pleasant duty to express sincere

appreciation and gratitude to my leader to the professor G. S. Pisarenko and to my associates N. A. Wenzel, A. L. Kvitke, E. S. Umanskiy and N. N. Cherny for valuable councils and the diverse aid, render/shown to me during the execution of this work.

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Introduction.

§1. General characteristic of method.

Apropos of the common/general/total idea of the method of the initial functions it is possible to express different points of view. Specifically, it can be considered as generalization widely utilized in the strength of materials of the method of the initial parameters. The latter is based on the application/use of the universal equation of the elastic line of beam, which we will write in the form

$$w(x) = L_1 w_0 + L_2 \theta_0 + L_3 M_0 + L_4 Q_0 + \Phi(x). \qquad (0.1)$$

Here through $\Phi(x)$ are designated the terms, which consider type of load on beam, and

$$L_1 = 1,\ L_2 = x,\ L_3 = \frac{x^2}{2EI},\ L_4 = \frac{x^3}{6EI}. \qquad (0.2)$$

Functions L_j (j = 1, 2, 3, 4) they can be named the operators, since as a result of their application/use to the initial parameters w_0, θ_0, M_0, Q_0 (simple multiplication by these parameters) is obtained the value of sagging/deflection in any section of beam.

Expression (0.1) it is the special form of the notation of the general solution of the differential equation of the elastic line

$$EI \cdot \frac{d^4 w}{dx^4} = q(x), \qquad (0.3)$$

in which are isolated the initial parameters.

If all parameters are known, then formula (0.1) gives the ready solution of problem. But if, as this usually is, is known the only part of the initial parameters, then, satisfying conditions for those sections, where the displacement/movements are known, are obtained the algebraic equations, from which are located the unknown initial parameters.

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Let us find now function w (x, y), satisfying on rectangle $0 \leqslant x \leqslant 1.0 \leqslant y \leqslant \lambda$ to certain differential equation, for example

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} = f(x, y). \qquad (0.4)$$

and to the specified boundary conditions of this rectangle. Similarly

(0.1) let us present the general solution of problem in the form

$$w(x, y) = L_1 w_0(y) + L_2 w_0'(y) + L_3 w_0''(y) + L_4 w_0'''(y) + L_{w f} f(x, y). \quad (0.5)$$

The functions

$$w_0 = w(0, y), \quad w_0' = \left.\frac{\partial w}{\partial x}\right|_{x=0} \quad \text{etc.} \quad (0.6)$$

are called the initial functions and play the same role, as the initial parameters in (0.1).

The operators L_j (j = 1, 2, 3, 4) they will take the form

$$L_j = A \operatorname{ch}\frac{\beta x}{\sqrt{2}} \cos\frac{\beta x}{\sqrt{2}} + B \operatorname{sh}\frac{\beta x}{\sqrt{2}} \sin\frac{\beta x}{\sqrt{2}} + C \operatorname{ch}\frac{\beta x}{\sqrt{2}} \sin\frac{\beta x}{\sqrt{2}} + D \operatorname{sh}\frac{\beta x}{\sqrt{2}} \cos\frac{\beta x}{\sqrt{2}}, \quad (0.7)$$

where β it indicates the symbol of the differentiation

$$\beta = \frac{d}{dy}. \quad (0.8)$$

Unlike (0.2) these values L_j are not usual functions, but they are transcendental linear differential operators. They completely are determined by equation (0.4) and by the taken form of general solution (0.5).

As concerns expression $L_{wf} f(x,y)$, that this be a particular solution to equation (0.4), that turns with x = 0 into zero together with its first-order three derivatives in terms of x.

If all the initial functions are known (Cauchy problem), then formula (0.5) gives the ready solution of problem, expressed in

symbolic form. In order to pass to the usual form of solution, it is necessary to know how to use operators L_i to the initial functions. For this purpose it is possible to use the tables of the values of the operators, whom it arrange/locates operational calculus.

But if is known the only part of the initial functions (boundary-value problem), then, by satisfying boundary conditions at edge $x = 1$, it is possible to obtain the system of transcendental differential equations for unknown initial functions. By integrating it taking into account boundary conditions on edges $y = 0$ and $y = \gamma$, let us find these functions and then on (0.5) we will obtain the solution of problem.

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A quantity of terms in expression (0.5) depends on the order of equation (0.4), the form of the notation of the expression itself does not depend on the initial differential equation or the conditions of problem. Therefore it is represented possible to call expressions of type (0.5) the canonical equations of the method of the initial functions.

Let us note still that instead of function w and its derivatives as the initial functions can be undertaked such linear combinations

cf these values, which are justified by the physical content of problem.

The described diagram of the application/use of a method of the initial functions (in more detail about this will go speech in chapter 3) shows that this method and the method of the initial parameters are completely analogous in project, but essentially they differ in the utilized by them apparatuses and the volumes of computational work. This is completely logical, since the method of the initial parameters deals with one-dimensional, and the method of the initial functions with two-dimensional problems.

As the basis of the method of the initial functions is placed the tendency:

a) to unify the process of the solution to any linear partial differential equation under the arbitrary right side of the equation and boundary or initial conditions (for this serves the introduction of canonical kononicheskikh equations of the method of the initial functions and the wide use of symbolic operations both for obtaining general solution and for the solution to specific problems);

b) to facilitate the solution of problems in that sense in order that the known part of work could be made previously and represented

in the form of the tables of some values and expressions (for this are located the "general solutions" of defined classes of the tasks of mathematical physics, are comprised table for the realization of operators and table of the roots of transcendental characteristic equations for the determined forms of boundary-value problems);

c) to obtain the general solution of problem in this form, which at least to a certain extent facilitated the analysis of solution from the viewpoint of the effect of boundary conditions and the right side of the differential equation.

The effectiveness of the method of the initial functions is determined to a considerable extent by the presence of the tables of "general solutions", the roots of characteristic equations and so on. Therefore most advisable is represented the application/use of a method in such questions, where it is necessary to solve a large quantity of problems, uniform in its content and differing only in terms of the boundary conditions, the size/dimensions of the range of change in the independent variables or in terms of the form of the function, which stands in the right side of the differential equation, and where, therefore, there is sense and possibility to conduct the mentioned tabulation.

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Such questions include, for example, the curvature, the oscillation/vibrations and the stability of plates, twisting prismatic rods, two-dimensional and axisymmetric problem of elasticity theory, two-dimensional stationary problem of thermal conductivity, plane and axisymmetric thermoelastic problems, plane and axisymmetric stationary thermoelastic problems, plane and axisymmetric the stationary problems of electrical and magnetic fields, two-dimensional and axisymmetric problem of the hydrodynamics of ideal fluid, etc.

As concerns the form of differential equation and range of change in the independent variables, it is logical, simpler anything are solved problems for equations with constant coefficients relative to the functions, determined in rectangular range; bulkier is obtained solution for the equations, written in curvilinear coordinates (with the being divided variables), relative to the functions, determined in the range, limited by coordinate lines. is possible the application/use of a method to equations and ranges of more general view (for example, equation with polynomial coefficients, the trapezoidal domain of function), but these questions are still developed very little.

Finally, the method of the initial functions, apparently, can be used to some theoretical studies of boundary-value problems, for example to the questions of existence and uniqueness of the solution of boundary-value problem, correctness of the setting of problem, estimation of a variation in the function in range with the assigned variation in the boundary conditions, etc.

However, this problem, strictly speaking, even are not placed.

§2. Survey/coverage of works, connected with the development of the method of the initial functions.

The method of the initial functions has comparatively short history and is connected mainly with works of A. I. Lur'ye and V. Z. Vlasova.

Into 1936 in the article [26] of A. I. Lur'ye it obtains the formulas, which express displacement/movements and the voltage/stresses at any point of the plate through displacement/movements u_0, v_0, w_0 points of medium plane and derived u'_0, v^1_0, w^1_0 of these displacement/movements over alternating/variable z (z axis is perpendicular to medium plane). To

these formulas it is possible to arrive, searching for the solution to the equations of Lame the theory of the elasticity

$$\Delta u + \nu \frac{\partial \vartheta}{\partial x} = 0, \ \Delta v + \nu \frac{\partial \vartheta}{\partial y} = 0, \ \Delta w + \nu \frac{\partial \vartheta}{\partial z} = 0 \qquad (0.9)$$

in the form of series according to degrees of z:

$$u = \sum_{s=0}^{\infty} z^{2s} u_{2s}(x, y) + \sum_{s=0}^{\infty} z^{2s+1} u'_{2s}(x, y) \text{ and etc.} \qquad (0.10)$$

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Further A. I. Lur'ye notes: "it is possible, however, and not to make the substitution of series (0.10 indicated) to equations (0.9), but to arrive at final results immediately, after using the following method of calculation. Let us introduce the designations

$$\frac{\partial}{\partial x} = d_1, \ \frac{\partial}{\partial y} = d_2, \ \frac{\partial}{\partial z} = (\ )', \ \Delta = d_1^2 + d_2^2. \qquad (0.11)$$

with which equation (0.9) they take the form

$$\begin{gathered} u'' + (\Delta + \nu d_1^2) u + \nu d_1 d_2 v + \nu d_1 w' = 0, \\ \nu d_1 d_2 u + v'' + (\Delta + \nu d_2^2) v + \nu d_2 w = 0, \\ \nu d_1 u' + \nu d_2 v' + (1 + \nu) w'' + \Delta w = 0. \end{gathered} \qquad (0.12)$$

By considering d_1, d_2 and Δ as constant numbers, let us integrate this system of three linear differential second order equations under the initial conditions: with z = 0

$$\begin{gathered} u = u_0, \ v = v_0, \ w = w_0, \\ u' = u'_0, \ v' = v'_0, \ w = w'_0. \end{gathered} \qquad (0.13)$$

As a result it is obtained

$$u = \cos z\sqrt{\Delta}\, u_0 + \frac{\sin z\sqrt{\Delta}}{\sqrt{\Delta}} u_0' - \frac{\nu}{2}\frac{z\sin z\sqrt{\Delta}}{\sqrt{\Delta}}\frac{\partial}{\partial x}\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} + w_0'\right) + \frac{\sin z\sqrt{\Delta}}{\sqrt{\Delta}} u_0' - \frac{\nu}{2(1+\nu)}\left(\frac{\sin z\sqrt{\Delta}}{\Delta\sqrt{\Delta}} - \frac{z\cos z\sqrt{\Delta}}{\Delta}\right)\frac{\partial}{\partial x}\left(\frac{\partial u_0'}{\partial x} + \frac{\partial v_0'}{\partial y} - \Delta w_0\right) \quad (0.14)$$

and so forth.

"It is obvious that, after replacing $\cos z\sqrt{\Delta}, \frac{\sin z\sqrt{\Delta}}{\sqrt{\Delta}}$ and so forth with their expansions in series according to degrees $z\sqrt{\Delta}$ and returning to value Δ, which, until now, it was considered as number, its value of the two-dimensional operator of Laplace, we will arrive at unknown series (0.10). At the same time we can thus far this not make, but operate directly with formulas (0.14) and return to power series already as a result of calculations".

Then the author applies the obtained expressions for displacement/movements and voltage/stresses to the analysis of the stressed state of plate with a thickness of h. Expanding transcendental terms in series according to degrees of h and being limited to the lowest degrees of h, it it comes to the following

equation of the flexure of the plate:

$$\Delta^2 w_0 + \frac{h^2}{10}\,\frac{8m-3}{m-1}\,\Delta^2 w_0 = \frac{1}{D}\,P_z. \qquad (0.15)$$

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This equation is proposed to solve according to the method successive approximation, representing w_0 in the form of a series according to degrees of h.

We so dwelt at A. I. Lur'ye's this work because in it were for the first time proposed two of the basic features of the method of the initial functions:

1) the unknown functions were represented in the form of the linear combination of the values of some operators (depending only on the form of differential equations) above functions, assigned on plane z = 0;

2) for obtaining general solution widely were utilized transcendental differential operations.

After six years, being occupied by the theory of thick plate/slabs, A. I. Lur'ye again returns to this question in the article [27]. After repeating those previously obtained by him

results, the author it transfer/converts to the question concerning the determination of six unknown functions u_0, v_0, w_0, u'_0, v'_0, w'_0.

Into 1941 N. A. Kil'chevskiys in its fundamental work [23] on the theory of shells it proposed to utilize surface conditions of equilibrium for obtaining the equations, determining static or geometric values on median surface.

Following the idea of N. A. Kil'chevskiy, by A. I. Lur'ye it obtains system of the form

$$\left.\begin{aligned} A_{11}u_0+A_{12}v_0+A_{13}w'_0=f_1(x,\ y),\\ A_{21}u_0+A_{22}v_0+A_{23}w'_0=f_2(x,\ y),\\ A_{31}u_0+A_{32}v_0+A_{33}w'_0=f_3(x,\ y).\end{aligned}\right\} \qquad (0.16)$$

where f_i - known functions, and A_{ik}- some differential operations of infinitely high order. For the solution of this system is introduced the resolving function $\Psi(x,y)$ with the aid of the relationship/ratios

$$u_0=\begin{vmatrix} A_{12} & A_{13}\\ A_{22} & A_{23}\end{vmatrix}\Psi;\ v_0=\begin{vmatrix} A_{13} & A_{11}\\ A_{23} & A_{21}\end{vmatrix}\Psi;\ w_0=\begin{vmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{vmatrix}\Psi \quad (0.17)$$

(if $f_1 = f_2 = 0$). Substitution to third equation (0.16) reduces to the equation

$$|A_{jk}|\Psi=F(\Delta)\Psi=f_3(x,\ y), \qquad (0.18)$$

where F - the integral function of operator Δ.

Further the author examines the sclutien tc equations (0.18). He indicates the method of obtaining the particular solution to equation (0.18), if f_3 (x, y) there is a polynomial from x and y or f (Δ) - polynomial from Δ. The determination of particular solution for the general case is not given.

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General solution of the homogeneous equations

$$F(\Delta)\Psi = 0 \tag{0.19}$$

is obtained in the form

$$\Psi = \sum_{k=1}^{\infty} \varphi_k, \tag{0.20}$$

where φ_k - the solution to the equations

$$(\Delta - \delta_k)\varphi_k = 0. \tag{0.21}$$

a δ_k - the roots of the transcendental equation

$$F(\delta) = 0 \tag{0.22}$$

(these roots there will be an infinite multitude).

Then the author examines in detail uniform solutions for cases $f_1 = f_2 = 0$, $f_2 = f_3 = 0$ and $f_3 = f_1 = 0$ and gives the containing arbitrary parameters of expression for displacement/movements and voltage/stresses in plate/slab at symmetrical and skew-symmetric

(relative to medium plane) loads. In this case large role plays the equation

$$1 \pm \frac{\sin x}{x} = 0. \qquad (0.23)$$

In the last/latter paragraph of article the obtained expressions are applied to the solution of the problem of the flexure of circular thick plate/slab.

Thus, in this deep in the content article of A. I. Lur'ye substantially developed proposed to them earlier method, after using for determining functions u_0, v_0, w_0, u'_0, v'_0, and w'_0 partial differential equations of infinitely high order and after indicating the method of obtaining the uniform solutions of these equations, and also particular solutions for some forms of operators and right sides of the equation.

A. I. Lur'ye's works in this direction found their completion in published by it into 1955 fundamental monograph [28]. Systematizing its previously obtained results, it considerably it deepened them and simultaneously enlarged the possibilities of the proposed method. Mainly this concerns:

1) the introduction of orthogonal coordinates in plane XY;

2) obtaining the particular solutions to transcendental

differential equation for the cases, when right side (load) is a polyharmonic function or the eigenfunction of the oscillation/vibrations of the diaphragm/membrane, when the function of load is represented by dual trigonometric series or Fourier integral for two variables, and also by a series fur'ye - Bessel for a circular range;

3) the examination of the thermal stresses in layer;

4) the solution of series of problems for compression and flexure of thick rectangular or circular plate/slab from static or thermal effects (in this case boundary conditions are satisfied in St. Venant sense). The method, close to this, which was applied in A. I. Lur'ye's early works, was successfully used Ya. F. Malkin (USA) [29] to the solution of the three-dimensional harmonic problem of the stationary temperature distribution in flat/plane plate. In this case Ya. F. Malkin does not use the transcendental form of the notation of the operators.

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Further works in this direction belong to V. Z. Vlasov and his school.

In that which was published in 1955 article [11] V. Z. Vlasov proceeds from the system of equations of the theory of the elasticity

$$\left.\begin{aligned}&\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}=0;\\&\tau_{xy}=\tau_{yx}=G\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right);\\&\sigma_x=\frac{2G}{1-2\nu}\left[(1-\nu)\frac{\partial u}{\partial x}+\nu\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right];\\&(x,\ y,\ z;\ u,\ v,\ w).\end{aligned}\right\}\quad(0.24)$$

Introducing the basic functions

$$\left.\begin{aligned}&U=Gu,\ V=Gv,\ W=Gw,\\&X=\tau_{xz},\ Y=\tau_{yz},\ Z=\sigma_z,\end{aligned}\right\}\quad(0.25)$$

it it searches for the general solution of equations (0.24) for the layer, limited by planes z = 0 and z = const, in the form of Maclaurin series in alternating/variable z

$$\left.\begin{aligned}&U=U_0+z\left(\frac{\partial U}{\partial z}\right)_0+\frac{z^2}{2!}\left(\frac{\partial^2U}{\partial z^2}\right)_0+\ldots,\\&V=V_0+z\left(\frac{\partial V}{\partial z}\right)_0+\frac{z^2}{2!}\left(\frac{\partial^2V}{\partial z^2}\right)_0+\ldots,\\&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\&Z=Z_0+z\left(\frac{\partial Z}{\partial z}\right)_0+\frac{z^2}{2!}\left(\frac{\partial^2Z}{\partial z^2}\right)_0+\ldots.\end{aligned}\right\}\quad(0.26)$$

As a result of exception/elimination from equations (0.24) of stresses σ_x, σ_y and $\tau_{xy}=\tau_{yx}$ is obtained the basic system of equations

$$\left.\begin{aligned}
&\frac{\partial U}{\partial z} = -\alpha W + X, \quad \frac{\partial V}{\partial z} = -\beta W + Y, \quad \frac{\partial Z}{\partial z} = -\alpha X - \beta Y,\\
&\frac{\partial X}{\partial z} = -\frac{1+\nu}{1-\nu}\alpha\beta V - \left(\beta^2 U + \frac{2}{1-\nu}\alpha^2 U\right) - \frac{\nu}{1-\nu}\alpha Z,\\
&\frac{\partial Y}{\partial z} = -\frac{1+\nu}{1-\nu}\alpha\beta U - \left(\alpha^2 V + \frac{2}{1-\nu}\beta^2 V\right) - \frac{\nu}{1-\nu}\beta Z,\\
&\frac{\partial W}{\partial z} = -\frac{\nu}{1-\nu}(\alpha U + \beta V) + \frac{1-2\nu}{2(1-\nu)} Z,
\end{aligned}\right\} \quad (0.27)$$

where markedly

$$\alpha = \frac{\partial}{\partial x}, \quad \beta = \frac{\partial}{\partial y}. \qquad (0.28)$$

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With the aid of (0.27) it is possible all derivatives $\left(\frac{\partial^n U}{\partial z^n}\right)_0, \ldots \left(\frac{\partial^n Z}{\partial z^n}\right)_0$ to express by differential operations α^m and β^m above the initial functions U_0, V_0, W_0, X_0, Y_0, Z_0. Then solution (0.26) is written in the form

$$\left.\begin{aligned}
&U = L_{uu}U_0 + L_{uv}V_0 + L_{uw}W_0 + L_{ux}X_0 + L_{uy}Y_0 + L_{uz}Z_0,\\
&V = L_{vu}U_0 + L_{vv}V_0 + L_{vw}W_0 + L_{vx}X_0 + L_{vy}Y_0 + L_{vz}Z_0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&L = L_{zu}U_0 + L_{zv}V_0 + L_{zw}W_0 + L_{zx}X_0 + L_{zy}Y_0 + L_{zz}Z_0,
\end{aligned}\right\} \quad (0.29)$$

where the operators Z_{ik} are represented by the infinite series

$$
\left.
\begin{aligned}
L_{uu} &= 1-\frac{z^2}{2}\left(\frac{2-\nu}{1-\nu}\alpha^2+\beta^2\right)+\frac{z^4\gamma^2}{24}\left(\frac{3-\nu}{1-\nu}\alpha^2+\right.\\
&\left.+\beta^2\right)-\frac{z^6\gamma^4}{720}\left(\frac{4-\nu}{1-\nu}\alpha^2+\beta^2\right)+\ldots\\
L_{uv} &= -\frac{z^2}{2(1-\nu)}\alpha\beta+\frac{z^4}{12(1-\nu)}\gamma^2\alpha\beta-\\
&-\frac{z^6}{240(1-\nu)}\gamma^4\alpha\beta+\ldots
\end{aligned}
\right\} \quad (0.30)
$$

and so forth, in which through γ is designated the differential operator

$$\gamma = \sqrt{\alpha^2+\beta^2} = \sqrt{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}}. \quad (0.31)$$

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Series (0.30) formally can be summed, and then:

$$
\begin{aligned}
L_{xx} &= L_{uu} = \cos\gamma z \frac{1}{2(1-\nu)}\frac{\alpha^2 z}{\gamma}\sin\gamma z,\\
L_{yx} &= L_{uv} = -\frac{1}{2(1-\nu)}\frac{\alpha\beta z}{\gamma}\sin\gamma z,\\
L_{zx} &= L_{uw} = -\frac{1}{2(1-\nu)}\frac{\alpha}{\gamma}\left[(1-2\nu)\sin\gamma z+\gamma z\cos\gamma z\right],\\
L_{wx} &= L_{uz} = -\frac{1}{4(1-\nu)}\frac{\alpha z}{\gamma}\sin\gamma z,\\
L_{vx} &= L_{uy} = -\frac{1}{4(1-\nu)}\frac{\alpha\beta}{\gamma^3}\left[\sin\gamma z-\gamma z\cos\gamma z\right).
\end{aligned}
$$

$$
\left.
\begin{aligned}
L_{ux} &= \frac{\sin\gamma z}{\gamma} - \frac{1}{4(1-\nu)}\frac{\alpha^2}{\gamma^2}\left(\sin\gamma z - \gamma z\cos\gamma z\right),\\
L_{uy} &= L_{vx} = -\frac{1}{2(1-\nu)}\frac{\beta}{\gamma}\left[(1-2\nu)\sin\gamma z + \gamma z\cos\gamma z\right],\\
L_{xy} &= L_{vu} = -\frac{1}{2(1-\nu)}\frac{\alpha\beta z}{\gamma}\sin\gamma z,\\
L_{uz} &= L_{wx} = \frac{1}{2(1-\nu)}\frac{\alpha}{\gamma}\left[(1-2\nu)\sin\gamma z - \gamma z\cos\gamma z\right],\\
L_{wy} &= L_{vz} = -\frac{1}{4(1-\nu)}\frac{\beta z}{\gamma}\sin\gamma z,\\
L_{yz} &= L_{wv} = \frac{1}{2(1-\nu)}\frac{\beta}{\gamma}\left[(1-2\nu)\sin\gamma z - \gamma z\cos\gamma z\right],\\
L_{xw} &= L_{zu} = \frac{1}{1-\nu}\alpha\gamma z\sin\gamma z,\\
L_{yy} &= L_{vv} = \cos\gamma z - \frac{1}{2(1-\nu)}\frac{\beta^2 z}{\gamma}\sin\gamma z,\\
L_{wz} &= \frac{1}{4(1-\nu)}\frac{1}{\gamma}\left[(3-4\nu)\sin\gamma z - \gamma z\cos\gamma z\right],\\
L_{xv} &= L_{yu} = -\frac{1}{1-\nu}\frac{\alpha\beta}{\gamma}\left(\nu\sin\gamma z + \gamma z\cos\gamma z\right),\\
L_{zz} &= L_{ww} = -\frac{1}{1-\nu}\gamma\left(\sin\gamma z - \gamma z\cos\gamma z\right),\\
L_{yv} &= -\frac{\alpha^2}{\gamma}\sin\gamma z - \frac{1}{1-\nu}\frac{\beta^2}{\gamma}\left(\sin\gamma z + \gamma z\cos\gamma z\right),\\
L_{xu} &= -\frac{\beta^2}{\gamma}\sin\gamma z - \frac{1}{1-\nu}\frac{\alpha^2}{\gamma}\left(\sin\gamma z + \gamma z\cos\gamma z\right),\\
L_{yw} &= L_{zv} = \frac{1}{1-\nu}\beta\gamma z\sin\gamma z,\\
L_{zw} &= -\frac{1}{1-\nu}\gamma\left(\sin\gamma z - \gamma z\cos\gamma z\right).
\end{aligned}
\right\} \quad (0.32)
$$

The remaining stresses are determined by the formulas

$$
\begin{aligned}
\sigma_x &= A_u U_0 + A_v V_0 + \ldots + A_z Z_0,\\
\sigma_y &= B_u U_0 + B_v V_0 + \ldots + B_z Z_0,\\
\tau_{xy} = \tau_{yx} &= C_u U_0 + C_v V_0 + \ldots + C_z Z_0.
\end{aligned}
\quad (0.33)
$$

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The operators A_1, B_1, C_1 they can be represented either in the form of the series

$$\left.\begin{aligned}
A_u &= \frac{2}{1-\nu}\alpha - \frac{z^2}{1-\nu}(2\alpha^2+\beta^2)\alpha + \\
&+ \frac{z^4}{12(1-\nu)}(3\alpha^2+\beta^2)\gamma^2\alpha - \ldots, \\
A_v &= \frac{2\nu}{1-\nu}\beta - \frac{z^2}{1-\nu}\Big[(1+\nu)\alpha^2 + \\
&+ \nu\beta^2\Big]\beta + \frac{z^4}{12(1-\nu)}\Big[(2+\nu)\alpha^2 + \\
&+ \nu\beta^2\Big]\gamma^2\beta - \ldots.
\end{aligned}\right\} \quad (0.34)$$

and of so forth or in the closed transcendent symbolic form

$$
\left.
\begin{aligned}
&A_u = \frac{2}{1-\nu}\alpha\cos\gamma z - \frac{z\alpha^3}{\gamma(1-\nu)}\sin\gamma z,\quad A_w = -\\
&\quad -\frac{\alpha^2+2\nu\beta^2}{(1-\nu)\gamma}\sin\gamma z - \frac{z\alpha^2}{1-\nu}\cos\gamma z,\\
&A_v = \frac{2\nu}{1-\nu}\beta\cos\gamma z - \frac{z\alpha^2\beta}{\gamma(1-\nu)}\sin\gamma z,\quad A_z =\\
&\quad = \frac{\nu}{1-\nu}\cos\gamma z - \frac{z}{2(1-\nu)\gamma}\alpha^2\sin\gamma z,\\
&A_x = \frac{z\alpha^3}{2(1-\nu)\gamma^2}\cos\gamma z + \frac{\alpha}{(1-\nu)\gamma}\left(2-\nu-\right.\\
&\quad \left.-\frac{\alpha^2}{3\gamma^2}\right)\sin\gamma z,\quad A_y = \frac{\beta}{\gamma(1-\nu)}\left(\nu-\frac{\alpha^2}{2\gamma^2}\right)\sin\gamma z +\\
&\quad + \frac{z\alpha^2\beta}{2(1-\nu)\gamma^2}\cos\gamma z;\\
&B_u = \frac{2\nu}{1-\nu}\alpha\cos\gamma z - \frac{z\alpha\beta^2}{(1-\nu)\gamma}\sin\gamma z,\\
&B_x = \frac{\alpha}{(1-\nu)\gamma}\left(\nu-\frac{\beta}{2\gamma^2}\right)\sin\gamma z + \frac{z\alpha\beta^2}{3(1-\nu)\gamma^3}\cos\gamma z,\\
&B_v = \frac{2\beta}{1-\nu}\cos\gamma z - \frac{z\beta^3}{(1-\nu)\gamma}\sin\gamma z,\quad B_y =\\
&\quad = \frac{\beta}{(1-\nu)\gamma}\left(2-\nu-\frac{\beta^2}{2\gamma^2}\right)\sin\gamma z + \frac{z\beta^3}{2(1-\nu)\gamma^2}\cos\gamma z,\\
&B_w = -\frac{2\nu\alpha^2-\beta^2}{(1-\nu)\gamma}\sin\gamma z - \frac{z\beta^2}{1-\nu}\cos\gamma z,\\
&B_z = \frac{\nu}{1-\nu}\cos\gamma z - \frac{z}{2(1-\nu)\gamma}\beta^2\sin\gamma z;\\
&C_u = \beta\cos\gamma z - \frac{z\alpha^2\beta}{(1-\nu)\gamma}\sin\gamma z,\quad C_x = \frac{\beta}{\gamma}\sin\gamma z -\\
&\quad -\frac{1}{3(1-\nu)}\frac{\alpha^2\beta}{\gamma^3}\left(\sin\gamma z - \gamma z\cos\gamma z\right),\\
&C_v = \alpha\cos\gamma z - \frac{z\alpha\beta^2}{(1-\nu)\gamma}\sin\gamma z,\\
&C_y = \frac{\alpha}{\gamma}\sin\gamma z - \frac{1}{2(1-\nu)}\frac{\alpha\beta^2}{\gamma^3}\left(\sin\gamma z - \gamma z\cos\gamma z\right),\\
&C_w = -\frac{\alpha\beta}{(1-\nu)\gamma}\left[(1-2\nu)\sin\gamma z + \gamma z\cos\gamma z\right],\\
&C_z = -\frac{\alpha\beta z}{2(1-\nu)\gamma}\sin\gamma z.
\end{aligned}
\right|
\qquad (0.35)
$$

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Further author it uses the obtained general solution of the equations of elasticity theory on the calculation of the plate/slabs of a constant thickness by 2 h, which is located according to the actions of symmetrical and skew-symmetric (relative to medium plane) loads, and also it indicates the possibility in principle of applying a method on the calculation of the plate/slab-shells of alternating/variable thickness. In this case for finding the initial functions it it uses the method, actually identical with the fact which it was used in work [27] and which it is expressed by equations (0.16) - (0.18).

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However, although V. Z. Vlasov it gives the transcendent form of the solution, expressed through the resolving function F (x, y):

for the symmetric loading

$$\left.\begin{aligned}U_0 &= \left[\frac{1-2\nu}{2(1-\nu)}\alpha\sin^2\gamma h - \frac{h\alpha\gamma}{2(1-\nu)}\sin\gamma h\cos\gamma h\right]F,\\ V_0 &= \left[\frac{1-2\nu}{2(1-\nu)}\beta\sin^2\gamma h - \frac{h\beta\gamma}{2(1-\nu)}\sin\gamma h\cos\gamma h\right]F,\end{aligned}\right\}$$

$$\left.\begin{aligned}Z_0 &= \left[\frac{\gamma^2}{1-\nu}\sin^2\gamma h + \frac{h\gamma^3}{1-\nu}\sin\gamma h\cos\gamma h\right]F,\\ &\frac{\gamma^2}{1-\nu}\sin\gamma h[\sin\gamma h\cos\gamma h+\gamma h]F + Z_h = 0;\end{aligned}\right\} \quad (0.36)$$

for a skew-symmetric load (bending of plate/slab)

$$\left.\begin{aligned}W_0 &= \left[-\cos^2\gamma h + \frac{\gamma h}{2(1-\nu)}\sin\gamma h\cos\gamma h\right]F,\\ X_0 &= \frac{h\alpha\gamma}{1-\nu}\sin\gamma h\cos\gamma hF,\\ Y_0 &= \frac{h\beta\gamma}{1-\nu}\sin\gamma h\cos\gamma hF,\\ &\frac{\gamma}{1-\nu}\cos\gamma h[\sin\gamma h\cos\gamma h-\gamma h]F - Z_h = 0.\end{aligned}\right\} \quad (0.37)$$

but it recommends for the practical solution to use to the truncated sums of expressions (0.30) and (0.34), being limited to 2-3 first terms. So that, for example, instead of (0.37) it will be

$$\left.\begin{aligned} W_0 &= \left[1 - \frac{h^2(3-2\nu)}{2(1-\nu)}\nabla^2 + \frac{h^4}{3}\frac{2-\nu}{1-\nu}\nabla^2\nabla^2 - \dots\right]F, \\ X_0 &= \left[-\frac{h^2}{1-\nu}\nabla^2 + \frac{2h^4}{3(1-\nu)}\nabla^2\nabla^2 - \dots\right]\alpha F, \\ Y_0 &= \left[-\frac{h^2}{1-\nu}\nabla^2 + \frac{2h^4}{3(1-\nu)}\nabla^2\nabla^2 - \dots\right]\beta F, \\ &\left[\frac{2h^3}{3(1-\nu)} - \frac{h^5}{2(1-\nu)}\nabla^2 + \frac{h^7}{8(1-\nu)}\nabla^2\nabla^2 - \right. \\ &\qquad \left. - \dots\right]\nabla^2\nabla^2 F = Z_h. \end{aligned}\right\} \quad (0.38)$$

In the work in question V. Z. Vlasov for the first time 1) it formulated the basic ideas of the method of the initial functions it gave name itself this method; 2) for the precise representation of the general solution of the equations of elasticity theory through the initial functions and the acting on them operators; 3) it selected as the initial functions of the values, making concrete/specific/actual sense in the theory of elasticity (displacement/movement and voltage/stresses on plane of reference); 4) it indicated the possibility of removal from the properties of the operators of some conclusions of general nature; so, from the equality of the operators

$$L_{xu} = L_{xu},\ L_{uv} = L_{yx}$$ and so forth (0.39)

it follows theorem from the reciprocity of works.

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In recently left book [12] of V. Z. Vlasov and N. N. Leont'yev along with the account of the material of article [11] given certain improvement of method, for example, instead of (0.36) obtained the simpler solution

$$\begin{aligned} U_0 &= \frac{\alpha}{2}\left[(1-2\nu)\frac{\sin\gamma h}{\gamma} - h\cos\gamma h\right]\Phi, \\ V_0 &= \frac{\beta}{2}\left[(1-2\nu)\frac{\sin\gamma h}{\gamma} - h\cos\gamma h\right]\Phi, \\ Z_0 &= \gamma(\sin\gamma h + \gamma h\cos\gamma h)\Phi, \\ \gamma^2 h&\left(1+\frac{\sin 2\gamma h}{2\gamma h}\right)\Phi = Z_h, \end{aligned} \tag{0.40}$$

are given the general solutions of the task of the deformation of elastic support under the load, applied to his surface, and contact task in the calculation of plate/slab on elastic support, and also the basic results of V. Z. Vlasov's works period 1958.

Thus, A. I. Lur'ye and V. Z. Vlasov they proposed and they developed the method of the initial functions as method of bringing the three-dimensional task of elasticity theory for the layer, limited by planes z = 0 and z = h, to certain two-dimensional task. As the basis of method placed the isolation/liberation in the solution of the initial functions, the presence of these functions from the equations, obtained as a result of the satisfaction of boundary conditions on the planes, which they limit a layer, and the more or less wide use of symbolic operations. In this case in the authors of method the basic operator he is the given in different designations two-dimensional operator of Laplace

$$\Delta_1 = \Delta = \nabla^2 = D^2 = \gamma^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \qquad (0.41)$$

Unfortunately, A. I. Lur'ye's completely valid observation [27, page 151] about the fact that "the set-forth method of the compilation of the particular solutions to the equations of the equilibrium of elasticity theory it can be used to the large number of boundary-value problems of mathematical physics in the case of the bodies, limited by two parallel planes", it not found its realization in the operations of the authors of method.

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Let us stress once again the differences in the author's approach to the method of initial functions:

1. A. I. Lur'ye it uses natural from mathematical point of view initial functions (function and their normal derivatives), but V. Z. Vlasov - by natural from a physical point of view functions (displacement/movement and voltage/stresses).

2. A. I. Lur'ye wider it uses transcendent operations and transcendental differential equations (thanks to which are obtained solution the accurately satisfying differential equation tasks and boundary conditions on two planes z = const), but is forced therefore to examine the only polyharmonic and other quotients, although sufficiently the wide, classes of loads. V. Z. Vlasov in principle it does not limit the generality of loads, but it uses the predominantly truncated sums of operators (0.30) and (0.34), as a result of which it it comes not to transcendent, but to polyharmonic differential equations. In this case it is obtained the solution, which accurately it satisfies only conditions on primary surface.

Each of the approaches it can have its advantages in the definite missions.

Let us note now two circumstances, characteristic for the works of both authors and which are to a certain extent a shortcoming in

the proposed by them method.

1. Limited use of the transcendent differential operators - in essence for the reduced recording of series. Hence naturally it escape/ensues identification by them the method of the initial functions with sequence method. That, for example, in A. I. Lur'ye [28, page 190] he tells that "the symbolic method of the recording of the solutions to the equations of theory of elasticity, of course, it is not the necessary means of the study of the problems, connected with the equilibrium of a layer. It would be possible from the very beginning the solution to these equations to seek in the form

$$\left.\begin{aligned} u &= U(z)\,e^{i(\alpha x+\beta y)}, \\ v &= V(z)\,e^{i(\alpha x+\beta y)}, \\ w &= W(z)\,e^{i(\alpha x+\beta y)}. \end{aligned}\right\} \qquad (0.42)$$

Symbolic method it made it possible simple economical to organize this calculation, prompting in each stage further motion of unpacking/facing". Scarcely whether it is possible to agree with this affirmation therefore, that, for example, for Cauchy's problem

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2},\ a = \text{const};\ 0 < x,\ t < +\infty, \\ &u(x,\,0) = \varphi(x);\ \left(\frac{\partial u}{\partial t}\right)_{t=0} = \psi(x);\ u(0,\,t) = 0 \end{aligned}\right\} \qquad (0.43)$$

through the method of the initial functions easily (see [2, page 1309]) it is located the solution of d' Alembert

$$u(x,\,t) = \frac{1}{2}\left[\varphi(x+at) + \varphi(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\,d\xi, \qquad (0.44)$$

while this obtaining similar to it solutions with the help of series extremely difficultly, but sometimes is generally impossible.

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2. The absence of any substantiation of the actions, produced above the transcendent operators, to say nothing of about the fact that are not shown the domain of definition of the operators (that, true, it is of purely theoretical interest). Thus, for instance, the formula

$$(\sin^2 z\sqrt{\Delta}+\cos^2 z\sqrt{\Delta})\,\varphi(x,\ y)=\varphi(x,\ y) \qquad (0.45)$$

obvious, if Δ be a number, and it requires not only proof, but also interpretation, if Δ is an operator of Laplace.

Following works of A. I. Lur'ye and V. Z. Vlasov in recent years it appeared a series of the articles, in which they were solved specific problems with the application/use of a method of the initial functions, they were developed the separate/individual positions of this method, they were given some of its generalizations.

So, into 1958 they left V. V. Vlasov's article [16, 14, 13], article E. S. Umanskiy, A. L. Kvitkya and V. A. Agareva [41] and

the article of Ye. I. Silkin and N. A. Solov'yev [38].

In the article [16] author it used the method of the initial functions to the static task of the equilibrium of diaphragm/membrane and plicated type moment-less slightly curved shell almost with any form cf loading.

In the article [41] the authors was obtained the general solution of the equations of the axisymmetrical task of elasticity theory in the presence of temperature and inertia actions, and also the equations of the torsion of body of revolution with the arbitrary form of axial section/cut.

Characteristic for both articles it is that here for the first time the method of the initial functions it was used for the reducing of two-dimensional tasks to ordinary differential equations. The difference between them (to say nothing, of course, about series of question touched upon) it consists in the following:

in V. V. Vlasov's work method it is used to comparatively idle time, from mathematical point of view, task, and therefore it is possible to obtain expressions for the operators in the closed transcendent form;

in work E. S. Umanskiy, A. L. Kvitkya and V. A. Agareva the task in question considerably more complex, but the operators are obtained only in the form of series (besides torsion).

In the article [14], using the method of the initial functions, V. V. Vlasov it obtains the general solution of the equations of two-dimensional problem of elasticity theory, whereupon the operators are expressed in the closed transcendent form. Then the general solution it is used to the tasks of the depression of die/stamp into plate/slab, about the stressed state of two-layered plate/slab, of the stability of three-layered plate/slab, to research of the voltage distributions in double T during load one flanges by distributed load, and also to the study of bending and torsion of thin-walled box.

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In all cases boundary conditions on two of the four opposite edges they correspond to the known solutions of Paylon and Rib'yer.

As concerns the process of obtaining general solution, in our opinion, author it allow/assumes excessive liberty in rotation with the operators (although to a certain extent this is generally been inherent in symbolic methods). Thus, for instance, it

record/writes system of equations for determining those entering the general solution of values C_k as follows:

$$\left.\begin{array}{l} -\frac{1+\mu}{2}\gamma(\gamma C_1+C_4)=u^{\circ},\ \gamma\left[\frac{1+\mu}{2}\gamma C_2+(1-\mu)C_3\right]=v^{\circ}, \\ \gamma^2 G[(1+\mu)\gamma C_1-(1-\mu)C_4]=\sigma^{\circ}_{y},\ \gamma^2 G[(1+\mu)\gamma C_2-2\mu C_3]=\tau^{\circ}_{xy}, \end{array}\right\} \quad (0.46)$$

whence by purely algebraic way it finds C_1, C_2, C_3, C_4, although here

$$\gamma=\frac{\partial}{\partial x}, \quad (0.47)$$

a value C_k they are functions of x.

To two-dimensional problem of elasticity theory is dedicated and V. V. Vlasov's candidate thesis [15].

In the article [13], on the basis of the obtained in [14] general solution of two-dimensional problem, author it gives the solution of the problem of the bending (in the sense of plane strain) of multilayer plate/slab - final, by periodically depending on and not limited; in this case again on two end/faces boundary conditions they correspond to the solutions of Faylon and Rib'yer.

Finally, in the article [38] Ye. I. Silkin and N. A. Solovyev, on the basis of equations (0.29), (0.32), (0.33) and (0.35) V. Z. Vlasov and using the method of double trigonometric series, they

trace the limits of the applicability Kirchoff - Love's hypothesis for thick square plate/slabs with hinged support on outline/contour.

In 1959--1961 they left the article of V. V. Vlasov [17, 18] and of article of the author of this work [1, 2], and also of the article, written by author together with N. A. Wenzel, E. S. Umanskiy and N. N. Chern [42, 10 and 3].

V. V. Vlasov's articles [17, 18] are dedicated to some particular questions of two-dimensional problem for a rectangular domain and the bending of rectangular plates. Unfortunately, in these articles author it does not give expressions for voltage/stresses and displacement/movements.

As concerns articles [1, 2, 3, 10, 41, and 42], in them is briefly illuminated part of the questions, presented in this book.

To this it is limited the enumeration of the works, direct-connected with the development of the method of the initial functions.

One should note, however, another the research, which do not have direct/straight relation to the method in question, but very close to it in its ideas.

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Most characteristic in this respect they are P. F. Papkovich's work [31, 32], I. Fadle [47] and G. A. Greenberg [19 and 20].

Into 1940 P. F. Papkovich [31] and I. Fadles [47] (latter - under the effect of the ideas, expressed by F. Tel'ke in work [50]) simultaneously and independently of each other used to two-dimensional problem of elasticity theory the method, which is actually the generalization of M. Levi's known method (for example, see [32, §14-24]) and that which consist of the fact that the unknown solution it is represented as expansion in terms of the functions, which are selected by special form, so that they they would satisfy some uniform conditions on two opposite boundaries of rectangular domain. In the article [20] G. A. Greenberg it generalized this method, after widening the class of functions, according to which it is conducted expansion/decomposition, and P. F. Papkovich [32] it spread this method to the tasks of the bending of rectangular plates. In specific problems P. F. Papkovich and I. Fadles they arrived at equation (0.23) they tabulated the first roots of this equation.

Into 1946 and then into 1948 [19], developing this tendency, G.

A. Greenberg it proposed for the sufficiently broad class of the tasks of mathematical physics, described by the differential second order equations, the method of the representation of solution in the form of Fourier series in the eigenfunctions of uniform task.

All the mentioned here works they have that that common/general/total with the method of the initial functions, that if, using the last/latter, transcendental functional equation is treated as ordinary differential equation of infinitely high order, then solution it is obtained that which was presented as expansion in terms of the fact eigenfunctions of uniform task. True, the coefficients of expansion they can be others, since P. F. Papkovich, I. Fadle and G. A. Greenberg select any particular solution, but in the method of the initial functions this solution it is obtained by completely determined - to the satisfying zero conditions on the initial line. This it clear Greenberg's method [19, 20], but it, as has already been spoken, was worked out only for second order equations.

In conclusion let us pause briefly at use in the method of the initial functions of the apparatus of operational calculus.

In §1 has already been noted that for the method of the initial functions the characteristically wide application of symbolic

operations. At first glance it can be shown that both for conducting operations themselves and for their proof they can be used apparatus and the theory of operational calculus. However, the situation is not entirely thus. Without going into details, let us note following.

1. In three-dimensional tasks the method of the initial functions it deals in essence with the flat/plane operator of Laplace, in operational calculation/enumeration this operator it does not figure.

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2. In two-dimensional tasks the basic operator, utilized in the method of the initial functions, he is differential operator, that in known sense it draws together this method with operational methods.

3. The purposes, as which they serve the operators in operational calculus and in the method of the initial functions, in essence are different:

a) in operational calculus - the presence of operator, who corresponds to the unknown solution, and then either this operator's realization for unit function (as is done in works [25, 35, 21 and 22], where operational calculus it is based on the integral transform

Laplace - Carson), or the presence of function, identical to the obtained operator (as is done in works [30] and [49], where operational calculus it is stated in purely operator form);

b) in the method of the initial functions - the presence of the operators, and then either their realization for the assigned/prescribed functions (in the case of the Cauchy problem), or the presence of unknown function from transcendent integrodifferential equations (in the case of boundary-value problem) during the wide use of symbolic operations.

4. During the solution to boundary-value problems for elliptical equations for the method of the initial functions dominant role they play the operators of the form

$$F(p) = e^{ikp} \text{ (k - real number)}, \qquad (0.48)$$

a in usual operational calculus such operators they are excluded, since the integral

$$\left|\int_{a-i\infty}^{a+i\infty} F(p)\,dp\right| < \int_{-\infty}^{+\infty} |e^{ik(a+i\sigma)}|\,d\sigma = \int_{-\infty}^{+\infty} e^{-k\sigma} d\sigma \qquad (0.49)$$

it diverges with any k and, it means (see for exam. [24, page 404]), F (p) not represented by Laplace's integral. Analogously and in operator calculation/enumeration Ya. Mikusinskiy "there does not

exist exponential function $e^{i\lambda s}$ (λ really)" [30, page 247].

5. From formal point of view those symbolic operations, which are used in the method of the initial functions, most of all they correspond to the style of works M. Vashchenko-Zakharchenk [9], O. Heaviside [48], V. I. Smirnova [39] and I. I. Khirshman and D. V. Uidder [45].

Thus, it is possible to do the following conclusions:

a) the used in the method of the initial functions symbolic operations they require the development of its "algebra" and its proof, not connected with the theory of operational (or operator) calculation/enumeration;

b) during the solution of the problems of Cauchy in the method of the initial functions they can be used the table of the images of operational calculus, in particular table from "Handbook" [22]; partially this it is related also to the presence of particular solutions (last/latter members in the canonical equations of the method of the initial functions).

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Chapter 1.

SOME QUESTIONS OF THE THEORY OF THE REGULAR AND SINGULAR OPERATORS.

In this chapter it is given certain proof by those to the transcendent differential operations, which widely are used in the method of the initial functions. It is natural that from the theory of the operators they are affected in essence those questions, which are connected with following presentation. However, incidentally are obtained and the separate/individual results, not connected with the method of the initial functions, but which are of to known degree common/general/total interest as, for example, integral relationships for Bessel functions, the particular solutions to harmonic and biharmonic equations, satisfying conditions Cauchy.

§3. Basic determinations.

Let $\varphi(\eta)$ be a function of dimensionless real or complex variable η.

(Let us note that subsequently it is necessary to deal predominantly with the interval/gap

$$0 \leqslant \eta \leqslant \lambda, \quad 0 < \lambda \leqslant 1). \qquad (1.1)$$

Let us introduce symbol β differentiation with respect to variable

$$\beta\varphi(\eta) = \frac{d\varphi}{d\eta} = \varphi'(\eta). \qquad (1.2)$$

Let, further, L (z) - analytic complex variable function z. If we in the analytical expression of this function replace z by β, then it will be obtained the formal expression L (β), which we call an operator. (Thus $|\beta|$, sign β, etc. are not the operators).

The fact that operator L (β) it corresponds (formally) to function L (z), it is noted by the sign $\leftrightarrow$:

$$L(\beta) \leftrightarrow L(z). \qquad (1.3)$$

If function L (z) besides alternating/variable z it contains even any parameter k, i. e., L (z) = L (k, z), then during replacement of z on β it is obtained operator L (k, β), depending on the parameter. The role of the parameter it can play also variable η or any another dimensionless and not depending on η and z variable ξ. In this case operators $L(\eta, \beta)$, L (ξ, β) or $L(\xi, \eta, \beta)$ they can be named operator-functions.

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Depending on the form of the function L (z) (whole, irrational, meromorphic, logarithmic and so forth) the corresponding name it is conferred and to operator. For example, L (β) = a = bβ^3 + cβ^6, L (β) = $\frac{a_1+\xi^2\beta^2}{a_2+\xi\beta}$, L ($\beta$) = $\sqrt{\beta^2+\nu^2}$. L (β) = sine m$\xi\beta$ they will be respectively polyncmial, fractional, irrational and trigonometric operator. Special place occupy the operators, assigned/prescribed not in the closed form, but in the form of any series, for example $\sum a_n z^n \leftrightarrow \sum a_n\beta^n$.

If to function $\varphi(\eta)$ it is used operator L (β) (otherwise: operator L it functions above the function ϕ or above the function ϕ it is produced operation L), then it is used any of the following recordings:

$$L(\beta)\{\varphi(\eta)\} = L(\beta)\varphi(\eta) = L\varphi(\eta) = L\{\varphi\} = L\varphi. \qquad (1.4)$$

As a result of action L above ϕ (if, of course, this action it makes sense) we obtain the function

$$\Phi(\eta) = L(\beta)\varphi(\eta), \qquad (1.5)$$

which it is called operator's value L above the function ϕ, or simply by operator's value. The process of finding operator's value let us name operator's realization. Sometimes so let us call formula itself, which it shows, which operations necessary to produce above ϕ in order to obtain Φ (for example, formula (1.2)).

Set ω_L all functions $\varphi(\eta)$, for which is possible operation (1.5), it is the domain of definition of operator L (β). Many all functions $\phi(\eta)$ above which are possible operation $L_1\varphi, L_2\varphi, \ldots, L_n\varphi$, obvious equal to product to set $\omega_{L_1}\omega_{L_2}\ldots\omega_{L_n}$. Set Ω_L the functions $\Phi(\eta)$, obtained as a result of applying operation (1.5) to all $\varphi(\eta)\in\omega_L$, it is called the range of values of operator L.

If monomial expression it contains operator L (β) and functions of η, then operator L (β) it functions only above those functions, which are placed to the right of it. For example, in expression $\varphi_1(\eta)L(\beta)\varphi_2(\eta)\varphi_3(\eta)$, operator it functions above the function $\psi(\eta)=\varphi_2(\eta)\varphi_3(\eta)$.

Now let us establish/install the elements of the algebra of the operators.

1. The operators are equal, i.e., $L_1 = L_2$, if:

a) $\omega_{L_1}=\omega_{L_2}$,

b) $L_1\varphi = L_2\varphi$ for any $\varphi\in\omega_{L_1}$. (1.6)

From $L_1 = L_2$ and $L_2 = L_3$, it follows $L_1 = L_3$.

2. Operator L (β) it is called unit it is designated β^0 or simply 1, if

$$L\varphi=\varphi \quad \text{for any} \quad \varphi\in\omega_1. \tag{1.7}$$

From the equality

$$L=\beta^0=1 \tag{1.8}$$

it follows

$$L\varphi = \beta^\circ\varphi = 1\{\varphi\} = \varphi. \quad (1.9)$$

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Obviously, ω_1 and Ω_1 coincide with many all functions.

3. Operator L (β) it is called zero (zero-operator) it is designated through 0, if

$L\varphi \equiv 0$ for any $\varphi \in \omega_0$. (1.10)

From the equality

$$L = 0 \quad (1.11)$$

it follows

$$L\varphi = 0\{\varphi\} \equiv 0. \quad (1.12)$$

Obviously, $\omega_0 = \omega_1$, and Ω_1 consists of one number zero.

4. Operator's multiplication by number (generally speaking, complex) it is determined by equality (see (1.5))

$$(cL)\varphi = c \cdot L\varphi = c\Phi(\eta). \quad (1.13)$$

This equality it will remain in force, if constant c is replaced by a function of some variable (including η). Obviously, $\omega_{cL} = \omega_L$.

5. Let

$$L_1(\beta)\varphi(\eta) = \Phi_1(\eta), \quad L_2(\beta)\varphi(\eta) = \Phi_2(\eta). \qquad (1.14)$$

Operator L it is called the sum of operators L_1 and L_2 it is designated

$$L = L_1 + L_2 = L_2 + L_1, \qquad (1.15)$$

if

$$L\varphi = \Phi_1 + \Phi_2 = \Phi_2 + \Phi_1. \qquad (1.16)$$

Obviously, $\omega_l = \omega_{L_1}\omega_{L_2}$. From the given determination it follows that the sum of the operators it is subordinated to communative and distributive laws whereupon for any (final) number of terms.

6. Let are as before accepted designation (1.14). Operator L it is called the product of the operators L_1 and L_2 it is designated

$$L = L(\beta) = L_1(\beta)L_2(\beta) = L_1L_2, \qquad (1.17)$$

if (under the conditions $\varphi \in \omega_{L_2}$, $\Omega_{L_2} \subset \omega_{L_1}$)

$$L\varphi = \Phi = L_1\Phi_2, \qquad (1.18)$$

or, in more detail,

$$L\varphi = L_1L_2\varphi = L_1(\beta)\{L_2(\beta)\varphi(\eta)\} = L_1(\beta)\Phi_2(\eta) = \Phi(\eta). \quad (1.19)$$

Analogously it is determined product L_2L_1. If moreover

$$L_1L_2\varphi = L_1\{L_2\varphi\} = L_2L_1\varphi = L_2\{L_1\varphi\}, \qquad (1.20)$$

that the operators L_1 and L_2 are permutable (are commutative).

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As concerns the domain of definition of operator L, scarcely possible to establish/install common/general/total dependence between ω_L, ω_{L_1} and ω_{L_2}. In the case of the adjustable operators the necessary (but, generally speaking, insufficient) condition it will be $\omega_L \subset \omega_{L_1}\omega_{L_2}$.

Not difficult to spread the given determination to product n of the operators and to show that this product it is subordinated to combined law, for example,

$$L_1L_2L_3L_4L_5 = (L_1L_2)(L_3L_4L_5) = (L_1L_2L_3)(L_4L_5) = \\ = L_1(L_2L_3)(L_4L_5). \qquad (1.21)$$

7. Operator, who is the product of the identical operators, it is called operator's degree:

$$LL = L^2, \quad LLL = L^3, \ldots, \quad \underbrace{LL\ldots L}_{n\ \text{раз}} = L^n. \qquad (1.22)$$

The commutativity of the identical degrees L is obvious, and from the associativity of multiplication it follows and the commutativity of the different degrees L. Hence: for any operator L with arbitrary entire m, n $\geqslant$ 0 it will be

$$L^{m+n} = L^mL^n = L^nL^m. \qquad (1.23)$$

From (1.2) and (1.22) we find the realization of degrees β:

$$\beta = \frac{d}{d\eta}, \quad \beta^2 = \frac{d^2}{d\eta^2}, \ldots, \quad \beta^n = \frac{d^n}{d\eta^n}, \ldots, \qquad (1.24)$$

from which we see that ~~otkuda vidnoy cto~~ $\omega_{\beta n} = \omega$, where through ω markedly many infinitely differentiated functions.

8. Operator L_1 (β) it is called reverse/inverse with respect to L (β) it is designated through L^{-1} (β) or $1/L$ (β), if it there exists and

$$L_1 L\varphi = L^{-1} L\varphi = \frac{1}{L}\cdot L\,(\varphi) = \varphi, \qquad (1.25)$$

i. e., (see (1.8) and (1.9))

$$L^{-1} L = \frac{1}{L}\cdot L = 1. \qquad (1.26)$$

If operators L and L^{-1} are permutable, then it is possible to record/write then:

$$LL^{-1} = L^{-1} L = L\cdot\frac{1}{L} = \frac{1}{L}\cdot L = \frac{L}{L} = 1, \qquad (1.27)$$

but if they not permutable, then multiply by inverse operator one should always to the left (this purely conditional understanding it escape/ensues from determination (1.25)), but recording L/L it becomes not determined.

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Generally in the case of the incommutable operators L_1 and $1/L_2$ fraction L_1/L_2 is not determined.

Analogous to (with 1.22) and (1.23) it is possible to obtain

$$\frac{1}{L^2} = L^{-2} = \frac{1}{L}\cdot\frac{1}{L}, \quad \frac{1}{L^n} = L^{-n} = \underbrace{\frac{1}{L}\cdot\frac{1}{L}\cdots\frac{1}{L}}_{n\ \text{раз}}. \qquad (1.28)$$

For an zero-operator (L = 0) inverse operator there does not exist. Operator, reverse/inverse unit (L = 1), unit itself (this it follows from (1.25) and p. 1), if $L^{-1} = 1$, then also L = 1.

9. As usual, operator L let us name linear (or additive), if

$$L\left\{\sum_{s=1}^{n} A_s\varphi_s\right\} = \sum_{s=1}^{n} A_s L\{\varphi_s\}, \qquad (1.29)$$

where A_s are constants or the functions of any alternating/variable independent variables of η. Hence we let us meet almost exclusively the linear operators.

In conclusion of this paragraph let us do several observations, useful for future reference.

1. Obvious that the equality

$$\varphi_1(\eta) = \varphi_2(\eta) \qquad (1.30)$$

it imply the equality

$$L\varphi_1 = L\varphi_2, \tag{1.31}$$

but reverse/inverse, generally speaking, erroneously.

2. From the equality

$$L_1(z) = L_2(z) \tag{1.32}$$

and the correspondences

$$L_1(\beta) \leftrightarrow L_1(z), \quad L_2(\beta) \leftrightarrow L_2(z) \tag{1.33}$$

it does not follow that

$$L_1(\beta) = L_2(\beta). \tag{1.34}$$

For example, let

$$L(z) = \frac{1}{1-z} = \begin{cases} L_1(z) = 1 + z + z^2 + \ldots, & \text{if} \quad |z| < 1, \\ L_2(z) = -\frac{1}{z} - \frac{1}{z^2} - \ldots, & \text{if} \quad |z| > 1. \end{cases} \tag{1.35}$$

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Then, after assuming that (1.32) and (1.33) it follows (1.34), we will have

$$\left.\begin{aligned} L(\beta) &= \frac{1}{1-\beta} = L_1(\beta) = 1 + \beta + \beta^2 + \ldots, \\ L(\beta) &= \frac{1}{1-\beta} = L_2(\beta) = -\frac{1}{\beta} - \frac{1}{\beta^2} - \ldots, \end{aligned}\right\} \tag{1.36}$$

whence in accordance with p. 1 page 27

$$1 + \beta^2 + \beta^3 + \ldots = -\frac{1}{\beta} - \frac{1}{\beta^2} - \frac{1}{\beta^3} - \ldots, \tag{1.37}$$

that erroneously. For this elimination and similar to it misunderstandings necessary to choose one of the possible expansions L (z) in a series (logically, after establish/installing preliminarily the criterion of this selection).

3. Let

$$L_1 = L_2, \quad L_1\varphi = \Phi_1, \quad L_2\varphi = \Phi_2. \tag{1.38}$$

According to p. 1 page 27,

$$L_1\varphi = L_2\varphi, \tag{1.39}$$

or

$$\Phi_1 = \Phi_2. \tag{1.40}$$

But then, according to p. 1 page 30,

$$L\Phi_1 = L\Phi_2, \tag{1.41}$$

either, taking into account (1.38),

$$L\{L_1\varphi\} = L\{L_2\varphi\}, \tag{1.42}$$

or, finally (see (1.19)),

$$LL_1\{\varphi\} = LL_2\{\varphi\}. \tag{1.43}$$

Hence again on p. 1 page 27

$$LL_1 = LL_2. \tag{1.44}$$

Analogously, on the basis of equality $L_1 = L_2$ and p. 1 page 27, it is possible to find

$$L_1L = L_2L. \tag{1.45}$$

Formulas (1.44) - (1.45) they show that the operator equation $L_1 = L_2$ it will not be broken, if it or to the right are multiplied to the right by the arbitrary operator L.

4. Let us designate

$$(L^{-1})^{-1} = L_1. \tag{1.46}$$

According to determination (1.26), if L_1 there exists, then

$$L_1 L^{-1} = 1. \tag{1.47}$$

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Let us multiply to the right this operator equality (in the right part of it stands unit operator) by L:

$$L_1 L^{-1} L = 1 \cdot L = L, \tag{1.48}$$

or, on the strength of the associativity of the product of the operators,

$$L_1 (L^{-1} L) = L, \tag{1.49}$$

i. e., (see (1.26))

$$L_1 = L. \tag{1.50}$$

Equate/comparing (1.46) and (1.50), we include that

$$(L^{-1})^{-1} = L. \tag{1.51}$$

Consequently, operator, is inverse with respect to inverse operator, is equal, if it there exists, to the initial operator.

After placing $L_2 = L^{-1}$ and using (1.26), easy to show that for

any L correctly expression (1.27), if only L^{-1} there exists.

5. If L^{-1} there exists, then it only.

6. Without the special labor it is possible to demonstrate that if we keep in mind the action of the arbitrary operators above the arbitrary $\varphi \in \omega_L$ Λ then: a) $L_1 + L_2 = L_1$ when and only when $L_2 = 0$; b) $L_1L_2 = L_2L_1 = 0$ when and only when and $L_1 = 0$, and $L_2 = 0$; c) $L_1L_2 = L_1$ then and only when $L_2 = 1$ (with $L_1 \neq 0$).

7. In all given above determinations and considerations the zero and unit operators they played the same role, as usual numbers zero and unity. Introduced for them designations 0 and 1 they stress that there is no need for making the difference between the operator or number.

8. If there exists L^{-1}, then of

$$LL_1 = LL_2 \text{ or } L_1L = L_2L \qquad (1.52)$$

it follows (p. 1, page 27)

$$L_1 = L_2, \qquad (1.53)$$

i. e., operator equality possible "to reduce" to the operator, which occupies identical extreme places in the left and right sides of the

equality. But if operator it stands not on identical places ($LL_1 = L_2L$) or not at all at edge ($L_1LL_2 = L_3LL_4$), then "cannot be reduced" (if, of course, operators L, L_1, L_2 they do not commutate either they do not occur of equality $L_1 = L_3$, or $L_2 = L_4$, then equations are led to expression (1.52)).

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§4. Exponential, hyperbolic and trigonometric operators.

In the literature (for example, [4, page 147, 5, page 10; 45, page 10]), frequently it is encountered operator $e^{k\beta}$. On the basis of formal expansion $e^{k\beta}$ according to degrees β and assuming that $\varphi \in \omega$, the authors easily they come to this operator's following realization: [1].

$$e^{k\beta}\varphi(\eta) = \varphi(\eta + k). \tag{1.54}$$

FOOTNOTE [1]. Here are given our designations. ENDFOOTNOTE.

Further I. I. Khirshman and D. V. Uidder they are limited to this observation: "now we define $e^{aD}f(x)$ as f (x + a) we note that in

spite of the background of definition, expression $e^{aD}f(x)$ it will make sense and when function f (x) is nct differentiated". This valuable by itself determination it does leave, however, opened this very important for a practice (at least in application to the method of the initial functions) question: actually whether of the fact that the operator in question they did designate $e^{k\beta}$. it was converted into exponential function and is it possible during operator transforms to use the properties of this function? In the present paragraph we let us attempt to give answer/response to this question, and also to a similar question for the trigonometric and hyperbolic operators.

Thus, let us determine operator-function $L_e(\xi,\beta)$ with the help of the equality

$$L_e(\xi,\beta)\,\varphi(\eta)=\varphi(\eta+\xi) \tag{1.55}$$

let us explain its some properties.

1. With $\xi = 0$ it will be

$$L_e(0,\beta)\,\varphi(\eta)=\varphi(\eta), \tag{1.56}$$

whence, according to p. 2 pages 27,

$$L_e(0,\beta)=1. \tag{1.57}$$

2. Set/assuming in (1.55) $\xi = \xi_1$ and $\xi = \xi_2$, we will obtain two different operators $L_e(\xi_1,\beta)$ and $L_e(\xi_2,\beta)$. For their product, taking into account (1.19) and (1.55), let us have:

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$$L_e(\xi_1,\beta)\,L_e(\xi_2,\beta)\,\varphi(\eta)=L_e(\xi_1,\beta)\ \{L_e(\xi_2,\beta)\,\varphi(\eta)\}=$$
$$=L_e(\xi_1,\beta)\,\varphi(\eta+\xi_2)=\varphi(\eta+\xi_2+\xi_1)=L_e(\xi_1+\xi_2,\beta)\,\varphi(\eta)\quad(1.58)$$

and, analogously,

$$L_e(\xi_2,\beta)\,L_e(\xi_1,\beta)\,\varphi(\eta)=L_e(\xi_2,\beta)\,\{L_e(\xi_1,\beta)\,\varphi(\eta)\}=$$
$$=L_e(\xi_1+\xi_2,\beta)\,\varphi(\eta).\quad(1.59)$$

From last/latter two equalities and p. 1 page 28 we obtain

$$L_e(\xi_1,\beta)\,L_e(\xi_2,\beta)=L_e(\xi_2,\beta)\,L_e(\xi_1,\beta)=L_e(\xi_1+\xi_2,\beta).\quad(1.60)$$

3. Set/assuming in (1.60) $\xi_1 = \xi$, $\xi_2 = -\xi$ and taking into account (1.57), we find

$$L_e(\xi,\beta)\,L_e(-\xi,\beta)=1,\quad(1.61)$$

whence (multiplying to the left by operator $L_e^{-1}(\xi,\beta)=\frac{1}{L_e(\xi,\beta)}$ and keeping in mind p. 8 page 29)

$$L_e(-\xi,\beta)=\frac{1}{L_e(\xi,\beta)}.\quad(1.62)$$

4. Since there is no fixed $\xi=\xi_0$, with which, for any function, $\varphi(\eta)$ would be $\varphi(\eta+\xi_0)=0$, on the basis of (1.55) and p 3, p 28, we come to the conclusion that with all ξ

$$L_e(\xi,\beta)\neq 0.\quad(1.63)$$

We will be distracted for a period of the fact that $L_e(\xi,\beta)$ - this function of operator, and let us look to it as for the usual function of ξ, in which is included certain parameter β. Equalities (1.57), (1.60), (1.62) and (1.63) they establish that function $L_e(\xi,\beta)$ in main properties it coincides with usual exponential function, in particular (1.60) it expresses the base property (theorem of summation), which is inherent in this and only this function (see for example, [5, p. 75]). Therefore it will natural designate $L_e(\xi,\beta)$ as e to degree ξ. However, into analytical expression for $L_e(\xi,\beta)$ it

must enter even the parameter β. Given above considerations they do not give any indications in the relation to, as precisely it enters β in $L_e(\xi,\beta)$ (parameter β generally in them it did not figure - it it served only recall about the fact that $L_e(\xi,\beta)$ from not simply function, but at the same time and operator). Therefore for $L_e(\xi,\beta)$ it would be possible to take any of the following designations: $e^{\xi\beta}$, $e^{\eta\xi}$, $e^{\xi\cdot\beta}$ etc. (such designations, as βe^{ξ} and $e^{\xi+\beta}$, do not match up, since they they contradict determinations §3). Into §6 examined the action of operator $L_e(\xi,\beta)$ above functions $\varphi(\eta)\in\omega$, as a result of which it will be discovered the naturalness of designation $e^{\xi\beta}$. In order not to make the difference in the designations of operator, used to any and to infinitely differentiated to functions, let us take and here

$$L_e(\xi,\beta)=e^{\xi\beta},$$

after which (1.54) and (1.55) they will give

$$L_e(\xi,\beta)\,\varphi(\eta)=e^{\xi\beta}\varphi(\eta)=\varphi(\eta+\xi). \tag{1.64}$$

Now let us consider operators

$$L_c(\xi,\beta)=\frac{1}{2}\,[L_e(\xi,\beta)+L_e(-\xi,\beta)],\quad L_s(\xi,\beta)=$$
$$=\left[\frac{1}{2}\,L_e(\xi,\beta)-L_e(-\xi,\beta)\right], \tag{1.65}$$

i.e. in accordance with (1.64) and p. 1 page 27

$$L_c(\xi,\beta)\,\varphi(\eta)=\frac{e^{\xi\beta}+e^{-\xi\beta}}{2}\,\varphi(\eta)=\frac{1}{2}\,[\varphi(\eta+\xi)+\varphi(\eta-\xi)], \tag{1.66}$$

$$L_s(\xi, \beta)\varphi(\eta) = \frac{e^{\xi B} - e^{-\xi B}}{2}\varphi(\eta) = \frac{1}{2}[\varphi(\eta+\xi) - \varphi(\eta-\xi)].$$

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1. With $\xi = 0$, according to p. 2 and 3 pages 28, it will be

$$L_c(0, \beta) = 1, \quad L_s(0, \beta) = 0. \quad (1.67)$$

2. Set/assuming in (1.66) $\xi = \xi_1$ and $\xi = \xi_2$, let us find

$$L_c(\xi_1, \beta) L_c(\xi_2, \beta)\varphi(\eta) = L_c(\xi_2, \beta) L_c(\xi_1, \beta)\varphi(\eta) = \\ = \frac{1}{4}[\varphi(\eta+\xi_2+\xi_1) + \varphi(\eta-\xi_2+\xi_1) + \\ + \varphi(\eta+\xi_2-\xi_1) + \varphi(\eta-\xi_2-\xi_1)], \quad (1.68)$$

$$L_s(\xi_1, \beta) L_s(\xi_2, \beta)\varphi(\eta) = L_s(\xi_2, \beta) L_s(\xi_1, \beta)\varphi(\eta) = \\ = \frac{1}{4}[\varphi(\eta+\xi_2+\xi_1) - \varphi(\eta-\xi_2+\xi_1) - \\ - \varphi(\eta+\xi_2-\xi_1) + \varphi(\eta-\xi_2-\xi_1)], \quad (1.69)$$

whence

$$L_c(\xi_1+\xi_2, \beta) + L_c(\xi_1-\xi_2, \beta) = 2L_c(\xi_1, \beta) L_c(\xi_2, \beta). \quad (1.70)$$

3. Store/adding up and subtracting (1.68) and (1.69), we will obtain

$$L_c(\xi_1 \pm \xi_2, \beta) = L_c(\xi_1, \beta) L_c(\xi_2, \beta) \pm L_s(\xi_1, \beta) L_s(\xi_2, \beta). \quad (1.71)$$

whence, in particular when $\xi_1 = \xi_2 = \xi$,

$$L_c^2(\xi, \beta) - L_s^2(\xi, \beta) = 1. \qquad (1.72)$$

After building, similarly (1.68) and (1.69), products $L_s(\xi_1, \beta) L_c(\xi_2, \beta)$ and $L_s(\xi_2, \beta) L_c(\xi_1, \beta)$, let us find

$$L_s(\xi_1 \pm \xi_2, \beta) = L_s(\xi_1, \beta) L_c(\xi_2, \beta) \pm L_c(\xi_1, \beta) L_s(\xi_2, \beta), \qquad (1.73)$$

$$L_s(2\xi, \beta) = 2L_s(\xi, \beta) L_c(\xi, \beta) \qquad (1.74)$$

and so forth.

4. Substituting in (1.65) ξ on $-\xi$, is discovered that

$$L_c(-\xi, \beta) = L_c(\xi, \beta), \quad L_s(-\xi, \beta) = -L_s(\xi, \beta). \qquad (1.75)$$

Repeating the considerations, given above for an operator $L_s(\xi, \beta)$ and, in particular, paying attention to the fact that equation (1.71) it is the indicial functional equation for a hyperbolic cosine (see [5, p. 75]), and formulas (1.67), (1.70) - (1.75) they coincide with the formulas of hyperbolic trigonometry, we include that formula (1.67) determine the hyperbolic operators

$$\left.\begin{aligned} L_c(\xi, \beta)\varphi(\eta) &= \operatorname{ch} \xi\beta\varphi(\eta) = \frac{1}{2}[\varphi(\eta+\xi) + \varphi(\eta-\xi)], \\ L_s(\xi, \beta)\varphi(\eta) &= \operatorname{sh} \xi\beta\varphi(\eta) = \frac{1}{2}[\varphi(\eta+\xi) - \varphi(\eta-\xi)], \end{aligned}\right\} \qquad (1.76)$$

a (1.67), (1.70) - (1.75) they express the usual relationships:

$$\left.\begin{array}{c}\operatorname{ch} 0=1, \quad \operatorname{sh} 0=0, \\ \operatorname{ch}(-\xi\beta)=\operatorname{ch}\xi\beta, \quad \operatorname{sh}(-\xi\beta)=-\operatorname{sh}\xi\beta, \\ \operatorname{ch}(\xi_1+\xi_2)\beta+\operatorname{ch}(\xi_1-\xi_2)\beta=2\operatorname{ch}\xi_1\beta\operatorname{ch}\xi_2\beta, \\ \operatorname{ch}(\xi_1\pm\xi_2)\beta=\operatorname{ch}\xi_1\beta\operatorname{ch}\xi_2\beta\pm\operatorname{sh}\xi_1\beta\operatorname{sh}\xi_2\beta, \\ \operatorname{sh}(\xi_1\pm\xi_2)\beta=\operatorname{sh}\xi_1\beta\operatorname{ch}\xi_2\beta\pm\operatorname{ch}\xi_1\beta\operatorname{sh}\xi_2\beta, \\ \operatorname{ch}^2\xi\beta-\operatorname{sh}^2\xi\beta=1, \\ \operatorname{sh}2\xi\beta=2\operatorname{sh}\xi\beta\operatorname{ch}\xi\beta\end{array}\right\} \quad (1.77)$$

and so forth.

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Literal so it is possible to show that the operators, introduced by the formulas

$$\left.\begin{array}{l}\cos\xi\beta\varphi(\eta)=\frac{1}{2}[\varphi(\eta+i\xi)+\varphi(\eta-i\xi)], \\ \sin\xi\beta\varphi(\eta)=\frac{1}{2i}[\varphi(\eta+i\xi)-\varphi(\eta-i\xi)],\end{array}\right\} \quad (1.78)$$

they satisfy the functional relationships of usual trigonometry:

$$\left.\begin{array}{c}\cos 0=1, \quad \sin 0=0, \\ \cos(-\xi\beta)=\cos\xi\beta, \quad \sin(-\xi\beta)=-\sin\xi\beta, \\ \cos(\xi_1\pm\xi_2)\beta=\cos\xi_1\beta\cos\xi_2\beta\mp\sin\xi_1\beta\sin\xi_2\beta, \\ \sin(\xi_1\pm\xi_2)\beta=\sin\xi_1\beta\cos\xi_2\beta\pm\cos\xi_1\beta\sin\xi_2\beta \\ \sin^2\xi\beta+\cos^2\xi\beta=1, \\ \sin 2\xi\beta=2\sin\xi\beta\cos\xi\beta\end{array}\right\} \quad (1.79)$$

and so forth. Furthermore, equate/comparing (1.76) and (1.79), we see that

$$\left.\begin{array}{ll}\operatorname{ch} i\xi\beta = \cos\xi\beta, & \operatorname{sh} i\xi\beta = i\sin\xi\beta,\\ \cos i\xi\beta = \operatorname{ch}\xi\beta, & \sin i\xi\beta = i\operatorname{sh}\xi\beta.\end{array}\right\} \quad (1.80)$$

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Using (1.76) - (1.78) and (1.80), it is not difficult to obtain the following formulas for the realization of the operators, who correspond to the functions of Acad./Academician. A. N. Krylov ([25, page 38]):

$$\left.\begin{gathered}
Y_1(\xi,\beta)\varphi(\eta)=\operatorname{ch}\xi\beta\cos\xi\beta\varphi(\eta)=\frac{1}{2}[\operatorname{ch}(1+i)\xi\beta+\\
+\operatorname{ch}(1-i)\xi\beta]\varphi(\eta)=\frac{1}{4}\{\varphi[\eta+(1+i)\xi]+\varphi[\eta-\\
-(1+i)\xi]+\varphi[\eta+(1-i)\xi]+\varphi[\eta-(1-i)\xi]\},\\
Y_2(\xi,\beta)\varphi(\eta)=\frac{1}{2}(\operatorname{ch}\xi\beta\sin\xi\beta+\operatorname{sh}\xi\beta\cos\xi\beta)\varphi(\eta)=\\
=\frac{1}{4}[(1-i)\operatorname{sh}(1+i)\xi\beta+(1+i)\operatorname{sh}(1-i)\xi\beta]\varphi(\eta)=\\
=\frac{1}{8}(1-i)\{\varphi[\eta+(1+i)\xi]-\varphi[\eta-(1+i)\xi]\}+\\
+(1+i)\{\varphi[\eta+(1-i)\xi]-\varphi(\eta-(1-i)\xi]\}),\\
Y_3(\xi,\beta)\varphi(\eta)=\frac{1}{2}\operatorname{sh}\xi\beta\sin\xi\beta\varphi(\eta)=\\
=\frac{1}{2}[\operatorname{ch}(1+i)\xi\beta-\operatorname{ch}(1-i)\xi\beta]\varphi(\eta)=\\
=\frac{1}{4}\{\varphi[\eta+(1+i)\xi]+\varphi[\eta-(1+i)\xi]-\\
-\varphi[\eta+(1-i)\xi]-\varphi[\eta-(1-i)\xi]\},\\
Y_4(\xi,\beta)\varphi(\eta)=\frac{1}{4}(\operatorname{ch}\xi\beta\sin\xi\beta-\\
-\operatorname{sh}\xi\beta\cos\xi\beta)\varphi(\eta)=-\frac{1}{8}[(1+i)\operatorname{sh}(1+\\
+i)\xi\beta+(1-i)\operatorname{sh}(1-i)\xi\beta]\varphi(\eta)=\\
=-\frac{1}{16}(1+i)\{\varphi[\eta+(1+i)\xi]-\varphi[\eta-\\
-(1+i)\xi]\}+(1-i)\{\varphi[\eta+(1-i)\xi]-\\
-\varphi[\eta-(1-i)\xi]\},
\end{gathered}\right\}\qquad(1.81)$$

and also to the exponential-trigonometric operators:

$$e^{m\xi\beta}\cos n\xi\beta\varphi(\eta)=\frac{1}{2}\{\varphi[\eta+(m+in)\xi]+\varphi[\eta+(m-in)\xi]\},$$
$$e^{m\xi\beta}\sin n\xi\beta\varphi(\eta)=\frac{1}{2i}\{\varphi[\eta+(m+in)\xi]-\varphi[\eta+(m-in)\xi]\}, \qquad (1.82)$$

where m and n - any numbers.

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One should do another observation that in the same way as in the region of real numbers hyperbolic functions are not periodical, so also the examined operators they are not periodic with that sense of the symbol β, which to it is given.

Thus it is proved that for the exponential operator $e^{\xi\beta}$, the hyperbolic operators ch $\xi\beta$ and sh $\xi\beta$ and the trigonometric operators cos $\xi\beta$ and sine $\xi\beta$ are valid all those functional relationship/ratios, that also for the appropriate usual functions of ξ (whereupon independent of form and character of function $\varphi(\eta)$).

§ 5. Regular operators.

1. Polynomial operators.

To function $L_n(z)=\sum_{s=0}^{n} a_s z^s$ corresponds the operator

$$L_n(\beta)=a_0+a_1\beta+\ldots+a_n\beta^n. \tag{1.83}$$

which logical to name polynomial. On the basis (1.13), p. 5 pages 28,

(1.24) and (1.9) we obtain this operator's realization:

$$\Phi_n(\eta) = L_n\varphi = a_0\varphi(\eta) + a_1\varphi'(\eta) + \ldots + a_n\varphi^{(n)}(\eta)\ (\varphi \in \omega_{L_n}), \quad (1.84)$$

whereupon $a_0, a_1, \ldots, a_n$ they can be constants or the functions of any variable (including η). The domain of definition of operator ω_{Ln} is the set n once of the differentiated functions. The theory of such operators is thoroughly developed in the book [5, § 38]. Therefore here we will note (without proof) only following [1]:

a) operator L_n linear (see (1.29));

b) for L_n are valid the usual summation rules, subtraction and multiplication, whereupon the operators L_n and L_m are commutative with any whole m, n $\geqslant$ 0; therefore with the operators can be turned as with usual polynomials, for example

$$L_2 = \beta^2 - 5\beta + 6 = L_1^{(1)}L_1^{(2)} = (\beta - 3)(\beta - 2);$$

c) occurs the identity

$$L_n(\beta + k)\varphi(\eta) = e^{-k\eta}L(\beta)\{e^{k\eta}\varphi(\eta)\}, \quad (1.85)$$

where k is the constant or not depending on η variable value;

d) if

$$L_n\varphi = 0 \text{ for any } \varphi \in \omega_{L_n}, \quad (1.86)$$

that

$$a_0 = a_1 = \ldots = a_n = 0 \quad (1.87)$$

and vice versa;

e) let Γ— be certain closed domain in complex plane (if η - complex variable) or certain closed interval on real axle/axis (if it is real); if with $\eta \in \Gamma$ will be $\varphi(\eta) \in \omega_{L_{n+1}}$, then the function $\Phi_n(\eta)$, determined by equality (1.84), is continuous and differentiated in Γ.

FOOTNOTE [1]. However, all this it is not difficult to obtain directly on the basis § 3, but for the purpose of reduction we this not do. ENDFOOTNOTE.

2. Determination of regular operator.

a) let us compose formal expression (power operational series)

$$\sum_{s=0}^{\infty} a_s\beta^s = a_0 + a_1\beta + a_2\beta^2 + a_3\beta^3 + \ldots. \qquad (1.88)$$

It determines the regular operator L (β) = L $^+$(z)

$$\sum_{s=0}^{\infty} a_s\beta^s = \lim_{n\to\infty} L_n(\beta) \sim L^+(\beta) = L(\beta) \leftrightarrow L^+(z) \qquad (1.89)$$

in that and only that case, if the series

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots \qquad (1.90)$$

converge to function $L^+(z)$ in an entire (final) plane complex variable z. In order to emphasize operator's regularity (if for this be a need), we agree to above place sign +.

Let us introduce the designation: $\omega^{(A)} \subset \omega$ [1] - many assigned on Γ functions $\varphi(\eta)$ such, that they are infinitely differentiated, whereupon

$$|\varphi^{(n)}(\eta)| < B \cdot A^n, \quad (\eta \in \Gamma), \qquad (1.91)$$

where A, B > 0 - constants.

FOOTNOTE [1]. See p. 7, page 29. ENDFOOTNOTE.

b) let us demonstrate following if power operational series (1.88) it determines regular operator (1.89), then the function series

$$a_0\varphi(\eta) + a_1\varphi'(\eta) + \dots + a_n\varphi^{(n)}(\eta) + \dots, \qquad (1.92)$$

obtained as a result of the formal application/use of operational

series (1.88) to functions $\varphi(\eta)\in\omega^{(A)}(\eta\in\Gamma)$, it converges and besides absolutely and evenly.

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Actually, on the strength of $\varphi\in\omega^{(A)}$ series (1.92) is majorized together

$$|a_0|+|a_1|A+|a_2|A^2+\ldots+|a_n|A^n+\ldots, \tag{1.93}$$

but power series (1.90) it converges in the entire plane z, a that means according to Abel's first theorem [40, page 49], it converges absolutely with $|z|=A$ (A - any). Therefore converges series (1.93). Then in Weierstrass's sign/criterion series (1.92) converges absolutely and evenly in closed domain Γ. (Let us note that the condition (1.91) can be replaced by less rigid). Let us designate the sum of this series by $\Phi_\infty(\eta)$:

$$\Phi_\infty(\eta)=a_0\varphi(\eta)+a_1\varphi'(\eta)+\ldots+a_n(\eta)^{(n)}(\eta)+\ldots(\varphi\in\omega^{(A)}). \tag{1.94}$$

From p. 2a (1.83), (1.94) and determination of summation of series, it follows that

$$\Phi_\infty(\eta)=\lim_{n\to\infty}\Phi_n(\eta)=\sum_{\mu=0}^{\infty}a_\mu\beta^\mu\varphi(\eta),(\varphi\in\omega^{(A)}). \tag{1.95}$$

c) according to the Weyerstrass theorem function $\Phi_\infty(\eta)$ is evenly continuous in closed domain Γ. Moreover, it is possible to show that

$$\Phi_\infty\in\omega^{(A)}. \tag{1.96}$$

Actually, the multiplication of converging series by which conveniently number does not change its radius of convergence.

Therefore the series

$$a_0 z + a_1 z^2 + a_2 z^3 + a_3 z^4 + \ldots \qquad (1.97)$$

will in an entire plane converge to function $zL^+(z)$. That means the expression

$$a_0 \beta + a_1 \beta^2 + a_2 \beta^3 + a_3 \beta^4 + \ldots \qquad (1.98)$$

determines regular operator $\beta L(\beta) \longleftrightarrow zL^+(z)$.

On the basis p. 2b the series

$$a_0 \varphi'(\eta) + a_1 \varphi''(\eta) + \ldots + a_n \varphi^{(n+1)}(\eta) + \ldots (\varphi \in \omega^{(A)}) \qquad (1.99)$$

will converge evenly (and absolutely) in domain Γ. But series (1.99) is obtained by term-by-term differentiation of series (1.94). Consequently [43, page 441],

$$\Phi'_\infty(\eta) = \lim_{n\to\infty} \Phi'_n(\eta) = a_0 \varphi'(\eta) + a_1 \varphi''(\eta) + \ldots (\varphi \in \omega^{(A)}). \qquad (1.100)$$

After using mathematic induction, let us find that with any whole m > 0

$$\Phi_\infty^{(m)}(\eta) = \lim_{n\to\infty} \Phi_n^{(m)}(\eta) = a_0 \varphi^{(m)}(\eta) + a_1 \varphi^{(m+1)}(\eta) + \ldots (\varphi \in \omega^{(A)}). \qquad (1.101)$$

i. e., $\Phi_\infty(\eta) \in \omega$.

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Now let us estimate the module/modulus of m-th derivative. Since $\varphi \in \omega^{(A)}$, that of (1.101) and (1.91) we find

$$|\Phi_\infty^{(m)}(\eta)| < |a_0||\varphi^{(m)}(\eta)| + |a_1||\varphi^{(m+1)}(\eta)| + |a_2||\varphi^{(m+2)}(\eta)| + \ldots <$$
$$< B[|a_0|A^m + |a_1|A^{m+1} + |a_2|A^{m+2} + \ldots] = BA^m[|a_0| +$$
$$+ |a_1|A + |a_2|A^2 + \ldots]. \qquad (1.102)$$

But representing whole function L (z) series (1.90) converges absolutely in any final part of plane z. Therefore

$$|a_0|+|a_1|A+|a_2|A^2+\ldots=C, \qquad (1.103)$$

where C > 0 - certain constant. Then, after designating BC = B_1, from (1.102) we will obtain

$$|\Phi_{\infty}^{(m)}(\eta)|<B_1A^m, \quad (\eta\in\Gamma), \qquad (1.104)$$

that, according to (1.91), and proves affirmation (1.96).

d) Arises the question concerning regular operator's realization: that whether operator's value will be function $\Phi_{\infty}(\eta)$ (see (1.94)), that whether

$$\Phi(\eta)=L^{+}(\beta)\varphi(\eta), \qquad (1.105)$$

where $L^{+}(\beta)$ it is determined from (1.89). Let us demonstrate that the result will be one and the same.

Preliminarily let us note that from the determination of operator (page 26 and p. 7 pages 32) and of the determination of zero operator (p. 3 pages 28) it follows that

$$0\longleftrightarrow L^{+}(z)\equiv 0, \qquad (1.106)$$

i.e. to zero operator corresponds $L^{+}(z)$ = 0, and, on the contrary, to function $L^{+}(z) \equiv 0$ correspond Lβ - 0.

Now let us introduce the designation

$$L_{\infty}(\beta)=\sum_{n=0}^{\infty}a_n\beta^n \qquad (1.107)$$

and subtract (1.95) from (1.102):

$$\Phi-\Phi_\infty=L^+\varphi-L_\infty\varphi=(L^+-L_\infty)\varphi\quad(\varphi\in\omega^{(A)}).\quad(1.108)$$

But

$$L^+(\beta)-L_\infty(\beta)\longleftrightarrow L^+(z)-\sum_{s=0}^{\infty}a_sz^s.\quad(1.109)$$

The last/latter expression is equal to zero, since $L^+(z)$ - this is summation of series $\sum_{s=0}^{\infty}a_sz^s$.

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Therefore $L^+(\beta)-L_\infty(\beta)\longleftrightarrow 0,$ or (On the strength of recently the done observation)

$$L^+(\beta)-L_\infty(\beta)=0.\quad(1.110)$$

Hence $L^+(\beta)=L_\infty(\beta)$, i.e.,

$$L^+(z)\longleftrightarrow L^+(\beta)=L_\infty(\beta)=a_0+a_1\beta+a_2\beta^2+\ldots=\lim_{n\to\infty}L_n(\beta).\quad(1.111)$$

Thus, in expression (1.89) sign ~ must be replaced by equal sign. Finally, from (1.100), (1.108) and (1.12) follows:

$$\Phi(\eta)=\Phi_\infty(\eta).\quad(1.112)$$

Thus, if $\varphi\in\omega^{(A)}$ and $\eta\in\Gamma$, then, taking into account (1.95) and (1.96),

$$L^+(\beta)\varphi=(a_0+a_1\beta+a_2\beta^2+\ldots)\varphi=a_0\varphi+a_1\varphi'+a_2\varphi''+\ldots=\Phi(\eta)\in\omega^{(A)},\quad(1.113)$$

i. e., any regular operator can be represented in two adequate forms: closed $L^+(\beta)$ and series $a_0+a_1\beta+a_2\beta^2+\ldots$

For example:

$$\left.\begin{aligned}
L^{+}(\beta) &= e^{m\beta} = 1 + m\beta + \frac{m^2}{2!}\beta^2 + \frac{m^3}{3!}\beta^3 + \ldots,\\
L^{+}(\beta) &= \operatorname{sh} m\beta = m\beta + \frac{m^3}{3!}\beta^3 + \frac{m^5}{5!}\beta^5 + \ldots,\\
L^{+}(\beta) &= \operatorname{ch} m\beta = 1 + \frac{m^2}{2!}\beta^2 + \frac{m^4}{4!}\beta^4 + \ldots,\\
L^{+}(\beta) &= \sin m\beta = m\beta - \frac{m^3}{3!}\beta^3 + \frac{m^5}{5!}\beta^5 - \ldots,\\
L^{+}(\beta) &= \cos m\beta = 1 - \frac{m^2}{2!}\beta^2 + \frac{m^4}{4!}\beta^4 - \ldots
\end{aligned}\right\} \quad (1.114)$$

All the examined previously concrete/specific/actual operators are special cases of common/general/total regular operator (1.113). So, with $a_0 = a_1 = \ldots = 0$ we will obtain the zero operator $L^{+}(\beta) = 0$; with $a_0 = 1$, $a_1 = a_2 = \ldots = 0$ - the single operator $L^{+}(\beta) = 1$; with $a_1 = 1$, $a_0 = a_2 = \ldots = 0$ - differential operator $L^{+}(\beta) = \beta$; with $a_1 \neq 0$, ..., $a_n \neq 0$, $a_{n+1} = a_{n+2} = \ldots = 0$ - polynomial operator.

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Let us note still that conformity (1.111) together with the proofs conducted establish/installs isomorphism between many whole complex variable functions z.

3. Properties of the regular operators. Differentiation with respect to symbol β.

a) on the strength of the linearity of the operation of differentiation any regular operator satisfies identity (1.29), i. e., it linear. Moreover

$$L^+\left\{\sum_{s=1}^{\infty} A_s\varphi_s(\eta)\right\} = \sum_{s=1}^{\infty} A_s L^+\varphi_s, \tag{1.115}$$

if in curly braces a series confronting converges evenly and absolutely, and $\varphi_s \in \omega^{(A)}$ and $\eta \in \Gamma$.

From linearity $L^+(\beta)$ it follows that

$$L^+(\beta)\{0\} = 0. \tag{1.116}$$

Therefore from $L^+(\beta)\varphi(\eta) = 0$ we consist that either $L^+(\beta) = 0$, or $\varphi(\eta) = 0$.

b) As the domain of definition of regular operator ω_{L^+}, as this escape/ensues from p. 2a, b, c and especially d, necessary to take set $\omega^{(A)}$ (page 39), i.e., $\omega_{L^+} = \omega^{(A)}$. Formulas (1.96), (1.112) and (1.113) show that then and $\Omega_{L^+} = \omega^{(A)}$. That means

$$\omega_{L^+} = \Omega_{L^+} = \omega^{(A)}, \quad \varphi \in \omega_{L^+}, \quad \Phi = L^+\varphi \in \Omega_{L^+}, \quad \eta \in \Gamma. \tag{1.117}$$

i.e. the application/use of a regular operator to set $\omega^{(A)}$ does not derive/conclude beyond the limits of this multitude.

Hence follows the existence of the product (p. 6 pages 28) of the finite number of regular operators - $L_1^+L_2^+\dots L_n^+$.

c) let $\varphi(\eta)\in\omega^{(A)}$, $\eta\in\Gamma$ and

$$L_1^+(\beta)=a_{10}+a_{11}\beta+a_{12}\beta^2+\dots,\quad L_2^+(\beta)=a_{20}+a_{21}\beta+a_{22}\beta^2+\dots; \tag{1.118}$$

$$\left.\begin{aligned}\Phi_1(\eta)=L_1^+(\beta)\varphi(\eta)=a_{10}\varphi(\varphi)+a_{11}\varphi'(\eta)+a_{12}\varphi''(\eta)+\dots,\\ \Phi_2(\eta)=L_2^+(\beta)\varphi(\eta)=a_{20}\varphi(\eta)+a_{21}\varphi'(\eta)+a_{22}\varphi''(\eta)+\dots\end{aligned}\right\} \tag{1.119}$$

According to p. 6. there exists

$$L_1^+L_2^+\varphi=L_1^+(\beta)\{L_2^+(\beta)\varphi(\eta)\}=(a_{10}+a_{11}\beta+\dots)\times \times\{a_{20}\varphi+a_{21}\varphi'+\dots\} \tag{1.120}$$

or, taking into account uniform and absolute convergence of series in (1.119),

$$L_1^+L_2^+\varphi=a_{10}a_{20}\varphi+(a_{10}a_{21}+a_{11}a_{21})\varphi'+\dots. \tag{1.121}$$

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After leading the same reasonings for a product $L_2^+L_1^+$, again we will obtain right side (1.21). Therefore

$$L_1^+L_2^+\varphi=L_2^+L_1^+\varphi=a_{10}a_{20}+(a_{10}a_{21}+a_{11}a_{20})\varphi'+ +(a_{10}a_{22}+a_{11}a_{21}+a_{12}a_{20})\varphi''+(a_{10}a_{23}+a_{11}a_{22}+a_{12}a_{21}+ +a_{13}a_{20})\varphi'''+\dots=\sum_{s=0}^{\infty}\Big(\sum_{l=0}^{s}a_{1,l}a_{2,s-l}\Big)\varphi^{(s)}. \tag{1.122}$$

Thus, two (but, and also, therefore, any finite number) regular operator are commutative:

$$L_1^+L_2^+\dots L_n^+=L_2^+L_1^+\dots L_n^+=L_n^+L_1^+\dots L_2^+=\dots. \tag{1.123}$$

for example,

$$\left.\begin{aligned}
&e^{m\beta}\sin k\beta = \sin k\beta\cdot e^{m\beta} = k\beta + mk\beta^2 + \\
&+\frac{k(3m^2-k^2)}{3!}\beta^3 + \frac{mk(m^2-k^2)}{3!}\beta^4 + \ldots, \\
&e^{m\beta}\cos k\beta = \cos k\beta\cdot e^{m\beta} = 1 + m\beta + \\
&+\frac{m^2-k^2}{2!}\beta^2 + \frac{m(m^2-3k^2)}{3!}\beta^3 + \\
&+\frac{m^4-6m^2k^2+k^4}{4!}\beta^4 + \ldots, \\
&e^{m\beta}\operatorname{sh} k\beta = \operatorname{sh} k\beta\cdot e^{m\beta} = k\beta + mk\beta^2 + \\
&+\frac{k(3m^2+k^2)}{3!}\beta^3 + \frac{mk(m^2+k^2)}{3!}\beta^4 + \ldots, \\
&e^{m\beta}\operatorname{ch} k\beta = \operatorname{ch} k\beta\cdot e^{m\beta} = 1 + m\beta + \\
&+\frac{m^2+k^2}{2!}\beta^2 + \frac{m(m^2+3k^2)}{3!}\beta^3 + \\
&+\frac{m^4+6m^2k^2+k^4}{4!}\beta^4 + \ldots, \\
&\beta^2 e^{-m\beta} = e^{-m\beta}\beta^2 = \beta^2 - m\beta^3 + \frac{m^2}{2!}\beta^4 - \frac{m^3}{3!}\beta^5 + \ldots.
\end{aligned}\right\} \quad (1.124)$$

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d) Unlike p. 6b, on the basis (1.123) and (1.116) we come to the conclusion that in order that the product of the regular operators would become zero, it is sufficient if at least one operator is zero (about the need for this condition see p. a).

e) On the basis p. a and c and uniform and absolute convergence of series in (1.113) we consist that series (1.88), which represent regular operators, can be store/added up, grouped and multiplied as polynomial operators. On the st[illegible]gth of identity (1.113) these

actions will be valid, also, for those closed expressions $L^{+}(\beta)$, which represent regular operators.

f) from p. c and e, and also p. 8 page 32, follows that any operational equality, which contains the only regular operators, will not be broken, if it are multiplied either are shortened by the regular operator $L^{+}(\beta)$ [1] or to both parts of the equality are adjoined on equal regular operator.

FOOTNOTE [1]. During reduction it is assumed that there is commutative $(L^{+})^{-1}$. ENDFOOTNOTE.

For example, if

$$L_1^2L_2+L_2+\beta L_3L_2=L_1L_2L_4+L_4, \qquad (1.125)$$

that are valid the conversions

$$L_1^2+\beta L_3=L_1L_4,$$
$$L_1(L_1-L_4)+\beta L_3=0 \text{ and so forth} \qquad (1.126)$$

It is clear that as in usual algebra, the operator, to which is conducted the reduction, must be different from zero.

g) On the basis of indicated in p. e, we shall conduct some multiplications and the additions of series (1.114).

After multiplying a series $e^{m\beta} = 1 + m\beta + \ldots$ to a series $e^{-m\beta} = 1 - m\beta + \ldots$, we will obtain unity. Therefore

$$e^{m\beta} \cdot e^{-m\beta} = 1. \tag{1.127}$$

Further, after multiplying a series $\sin m\beta = m\beta - \frac{m^3}{3!}\beta^3 + \ldots$ itself by itself, then after squaring a series $\cos m\beta = 1 - \frac{m^2}{2!}\beta^2 + \ldots$ and after forming the results, again we will obtain unity, i.e.,

$$\sin^2 m\beta + \cos^2 m\beta = 1. \tag{1.128}$$

That means the operators $e^{\pm m\beta} \longleftrightarrow e^{\pm mz}$, $\sin m\beta \longleftrightarrow \sin mz$ and $\cos m\beta \longleftrightarrow \cos mz$ satisfy thereby to functional relationship/ratios, as function $L^{+}(z)$, which correspond to these operators.

On the basis of the conformity (1.111) and the demonstrated above properties of the operators $L^{+}(\beta)$ it is possible to formulate the more common/general/total affirmation: if integral functions $L^{+}_{(1)}(z)$, $L^{+}_{(2)}(z)$, ..., $L^{+}_{n}(z)$ [2] satisfy certain functional relationship/ratio, which does not derive/conclude beyond the limits of many integral functions, then this same relationship/ratio they satisfy regular operators $L^{+}_{1}(\beta) \longleftrightarrow L^{+}_{1}(z)$, $L^{+}_{2}(\beta) \longleftrightarrow L^{+}_{2}(z)$, ..., $L^{+}_{n}(\beta) \longleftrightarrow L^{+}_{n}(z)$. For example, $(e^{\beta})^2 = e^{2\beta}$, $\sin 2\beta = 2\sin\beta\cos\beta$, $\sin k\beta = \frac{e^{k\beta} + e^{-k\beta}}{2i}$ etc.

FOOTNOTE [2]. Indices are undertaked into brackets in order not to mix these integral functions with polynomials 1st, 2nd and the so forth

cf degree. ENDFOOTNOTE.

h). Now let us introduce the concept of "differentiation" of $L^+(\beta)$ with respect to β. Let

$$L^+(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots \quad (1.129)$$

be integral function. It determines the regular operator

$$L^+(z) \longleftrightarrow L^+(\beta) = a_0 + a_1\beta + a_2\beta^2 + a_3\beta^3 + \ldots. \quad (1.130)$$

But $L^+(z)$ is infinitely differentiated. Therefore

$$\frac{dL^+(z)}{dz} = a_1 + 2a_2 z + 3a_3 z^2 + \ldots \quad (1.131)$$

also it will be integral function. As such, it determines the regular operator

$$\frac{dL^+(z)}{dz} \longleftrightarrow L^+_{(1)}(\beta) = a_1 + 2a_2\beta + 3a_3\beta^2 + \ldots. \quad (1.132)$$

Thus we will obtain

$$\frac{d^2L^+(z)}{dz^2} \longleftrightarrow L_{(2)}(\beta) = 2a_2 + 6a_3\beta + \ldots \quad (1.133)$$

and so forth.

Since, in the first place, series in (1.132), (1.133) and so forth can be obtained by formal differentiation of series (1.130), but in the second place, even on page 26 it was accepted on both sides of the sign $\longleftrightarrow$ to

write identical expressions (with replacement of z on β), let us agree instead of $L^+_{(1)}(\beta)$, $L^+_{(2)}(\beta)$ and so forth to write

$$\left.\begin{aligned}\frac{dL^+(\beta)}{d\beta} &= a_1 + 2a_2\beta + 3a_3\beta^2 + \ldots,\\ \frac{d^2L^+(\beta)}{d\beta^2} &= 2a_2 + 6a_3\beta + \ldots\end{aligned}\right\} \quad (1.134)$$

and to speak about differentiation (not limited) of operator $L^+(\beta)$.

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With this understanding in formulated in p. 9 affirmation of the functional relationship/ratios of the operators in the number of permissible actions, besides addition and multiplication, it is possible to include/connect differentiation with respect to symbol β. For example,

$$\frac{d\sin m\beta}{d\beta} = m\cos m\beta \quad (1.135)$$

(of this not difficult to be convinced it differentiated on β in (1.114) a series for sine $m\beta$ and after comparing result with together for cos $m\beta$).

More that, it is possible to indicate that the regular operator $L^+(\beta)$ satisfies certain "differential" equation or that it is the general solution of differential equation, implying by this, which to the corresponding differential equation satisfies $L^+(z) \leftrightarrow L^+(\beta)$.

For example, the operator

$$L(\beta) = A\sin\beta + B\cos\beta + \frac{1}{5}e^{2\beta}, \quad (1.136)$$

(A and B - arbitrary numbers) is "general solution" of the equation

$$\frac{d^2L}{d\beta^2}+L=e^{2\beta}, \qquad (1.137)$$

i.e.

$$L(z)=A\sin z+B\cos z+\frac{1}{5}e^{2z}\longleftrightarrow L(\beta)$$

- this is the general solution of the equation

$$L''+L=e^{2z}. \qquad (1.138)$$

This freedom in terminology is very convenient in appendices (just as designation e^{β}, sine $m\beta$ and so forth or the expression: "is decomposed by $L^{+}(\beta)$ in a series according to degrees β", "let us find summation of series $\beta+\frac{1}{3!}\beta^3+\ldots$»). However, one should remember that in actuality even the expression itself e^{β} does not have a sense, that this the only useful symbol, but all the mentioned above terms and "operations" only show, which actions must be produced in many functions $L^{+}(z)$ in order to obtain the function, which corresponds to the necessary operator. rightly to the existence of this terminology it is given by determinations and proofs p. 2 of present paragraph.

4. Regular operator-functions.

a) Let us consider now that the coefficients of series (1.90)

are the functions of variable ξ, that passes the interval/gap

$$0<\xi<1. \tag{1.139}$$

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Then regular operator and the corresponding to it integral function they will contain ξ as the parameter [1]:

$$L(\xi,\beta)=a_0(\xi)+a_1(\xi)\beta+a_2(\xi)\beta^2+\ldots, \tag{1.140}$$

$$L(\xi,\beta)\longleftrightarrow L(\xi,z)=a_0(\xi)+a_1(\xi)z+a_2(\xi)z^2+\ldots \tag{1.141}$$

FOOTNOTE [1]. The superscript +, which stresses operator's regularity L, for reduction let us lower. ENDFOOTNOTE.

The radius of convergence of a last/latter series must be equal to ∞ (p. 2a). Therefore functions $a_s(\xi)$ must be those which were limited with any s and ξ and satisfy condition [24, page 63]

$$\overline{\lim_{s\to\infty}}\sqrt[s]{|a_s(\xi)|}=0,\quad 0<\xi<1. \tag{1.142}$$

If it is implemented, then, obviously, all results of point/items 2-4 are spread also to operator-function. In particular (see (1.113))

$$L(\xi,\beta)\varphi(\eta)=[a_0(\xi)+a_1(\xi)\beta+a_2(\xi)\beta^2+\ldots]\varphi(\eta)=$$
$$=a_0(\xi)\varphi(\eta)+a_1(\xi)\varphi'(\eta)+a_2(\xi)\varphi''(\eta)+\ldots=\Phi(\xi,\eta). \tag{1.143}$$

Thus, the value of operator-function Φ will be the function of two

variables. As function η it will belong $\omega^{(A)}$ with any ξ of (1.139), if only $\varphi(\eta)\in\omega^{(A)}$. As concerns behavior Φ depending on ξ, it it is determined by the character of functions $a_s(\xi)$.

First of all let us show that if $a_s(\xi)$ are limited and satisfy condition (1.142), then series (1.143) it converges evenly relatively ξ. Actually, if are satisfied the conditions indicated, then function (1.140) whole and series (1.143) converges evenly relatively η (p. 2b) with any ξ. That means it is possible to indicate this natural number n_0, which does not depend on η and ξ, that with arbitrary m = 1, 2, 3, ... and $n \geqslant n_0$ occurs

$$|\Phi_{n+m}(\xi,\eta)-\Phi_n(\xi,\eta)| = |L_{n+m}(\xi,\beta)\varphi(\eta)-L_n(\xi,\beta)\varphi(\eta)|<\varepsilon. \tag{1.144}$$

If we here fix $\eta(\eta=\eta_0\in\Gamma)$, then (1.114) will express the necessary and sufficient condition (Bol'tsano - Cauchy) uniform relative to ξ the convergence of series (1.143) and of uniform in ξ continuity function $\Phi(\xi,\eta)$ in interval/gap (1.139).

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b) let us amplify now the limitations, superimposed during function $a_s(\xi)$. Specifically,, let us require, in crder to

$$|a_n'(\xi)|<M=\mathrm{const},\quad \overline{\lim_{n\to\infty}}\sqrt[n]{|a_n'(\xi)|}=0.\quad 0<\xi<1. \tag{1.145}$$

Then in the expression

$$\frac{\partial L(\xi,z)}{\partial\xi}=a_0'(\xi)+a_1'(\xi)z+a_2'(\xi)z^2+\ldots \tag{1.146}$$

a series is absolutely and evenly to converge in any final part of plane z and, thus, to determine whole function $\frac{\partial L(\xi;z)}{\partial\xi}$ by alternating/variable z, to which corresponds the regular operator

$$\frac{\partial L(\xi,z)}{\partial\xi}\longleftrightarrow L_{(1)}(\xi,\beta)-a_0'(\xi)+a_1'(\xi)\beta+a_2'(\xi)\beta^2+\ldots. \quad (1.147)$$

Consequently, the series

$$L_{(1)}(\xi,\beta)\varphi(\eta)-a_0(\xi)\varphi(\eta)+a_1'(\xi)\varphi^1(\eta)+a_2'(\xi)\varphi''(\eta)+\ldots \quad (1.148)$$

will be absolute and evenly (relatively η) to converge. By the literal repetition of the given above reasonings it is possible to show that this series converges evenly and relatively ξ from (1.139). Introducing now according to conformity (1.147) designation L_1 (ξ, β) and comparing series (1.143) and (1.148), on the basis of the Weyerstrass theorem we consist that series (1.143) allow/assumes term-by-term differentiation with respect to ξ and

$$\frac{dL(\xi,\beta)}{d\xi}\varphi(\eta)=[a_0'(\xi)+a_1'(\xi)\beta+a_2'(\xi)\beta^2+\ldots]\varphi(\eta)-$$
$$-a_0'(\xi)\varphi(\eta)+a_1'(\xi)\varphi'(\eta)+a_1'(\xi)\varphi''(\eta)+\ldots-\frac{\partial\Phi(\xi,\eta)}{\partial\xi}. \quad (1.149)$$

After superimposing during function $a_s(\xi)$ the even more rigorous conditions:

$$|a_s^{(m)}(\xi)|<M-\text{const},\quad \overline{\lim_{s\to\infty}}\sqrt[s]{|a_s^{(m)}(\xi)|}-0,\quad 0<\xi<1,\ m-1,2,\ldots, \quad (1.150)$$

let us arrive at

$$\frac{d^mL(\xi,\beta)}{d\xi^m}\varphi(\eta)-[a_0^{(m)}(\xi)+a_1^{(m)}(\xi)\beta+a_2^{(m)}(\xi)\beta^2+\ldots]\varphi(\eta)-$$
$$-a_0^{(m)}(\xi)\varphi(\eta)+a_1^{(m)}(\xi)\varphi'(\eta)+a_2^{(m)}(\xi)\varphi''(\eta)+\ldots-\frac{\partial^m\Phi(\xi,\eta)}{\partial\xi^m}\in\omega^{(A)}, \quad (1.151)$$

whereupon this series it converges absolutely and evenly (relatively ξ and η), if

$$0<\xi<1, \quad \eta\in\Gamma, \quad \varphi(\eta)\in\omega^{(A)}, \qquad (1.152)$$

a, according to (1.96), operator's value belongs to set $\omega^{(A)}$.

Furthermore, formula (1.153) shows that $\frac{d^m L(\xi,\beta)}{d\xi^m}$ it is regular operator-function.

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c) expression dL (ξ, β) /dξ is similar dL/dβ, it was introduced in p. b simply as symbol of operator, corresponding to function $\frac{\partial L(\xi,z)}{\partial z}$. It is possible, however, to show that dL/dξ, unlike dL/dβ, actually is derivative.

For this purpose let us give variable ξ increase h. Instead of (1.140) then we will obtain

$$L(\xi+h,\beta)=a_0(\xi+h)+a_1(\xi+h)\beta+a_2(\xi+h)\beta^2+\ldots \qquad (1.153)$$

Since operational series (1.140) and (1.153) converge absolutely (recall that this expression indicates absolute convergence of series (1.141), and also, therefore, series in (1.143), it is possible to piecemeal store/add up them (algebraically). By deducting (1.140) from (1.153), let us arrive at the new regular operator

$$\Delta_h L(\xi,\beta)=L(\xi+h,\beta)-L(\xi,\beta)=\sum_{n=0}^{\infty}[a_n(\xi+h,\beta)-a_n(\xi,\beta)]\beta^n. \qquad (1.154)$$

which logical to name an increase in the operator-function.

Let us assume that the functions of $a_s(\xi)$ satisfy condition (1.145). Then according to the law of mean (Lagrange) it will be

$$a_s(\xi+h,\beta)-a_s(\xi,\beta)=ha_s'(\xi'),\quad \xi<\xi'<\xi+h. \qquad (1.155)$$

That means

$$\Delta_h L(\xi,\beta)=hL_{(1)}(\xi',\beta), \qquad (1.156)$$

where $L_{(1)}(\xi',\beta)$ is given by formula (1.147).

Hence, by the way it is apparent that operator function L (ξ, β) is continuous in terms of variable ξ (whereupon is evenly continuous):

$$\lim_{h\to 0}\Delta_h L(\xi,\beta)=\lim_{h\to 0}[hL_{(1)}(\xi',\beta)]=0\cdot L_{(1)}(\xi,\beta)=0. \qquad (1.157)$$

After dividing (1.156) into h and after passing to limit, let us ascertain that $L_{(1)}(\xi,\beta)$ - this really/actually derivative of operator-function L (ξ, β) in terms of ξ:

$$L_{(1)}(\xi,\beta)=\lim_{h\to 0}\frac{\Delta_h L(\xi,\beta)}{h}=\frac{dL(\xi,\beta)}{d\xi}=a_0'(\xi)+a_1'(\xi)\beta+\ldots. \qquad (1.158)$$

Thus it is possible to show that i in (1.151) figures the usual m-th derivative.

If $a_s = \text{const}$, i.e., L (ξ, β) = L (β), then of (1.158) we obtain

$$\frac{dL(\beta)}{d\xi} = 0. \tag{1.159}$$

Therefore during differentiation of operator-functions L (ξ, β) the usual operators L (β) play the role of constants.

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Let us note that strictly the constants (number) on the strength of determination (1.8) also are related to L, for example, 5 = 5•1 = $5\beta^0$ = L (β).

On the basis of determination (1.58), it is possible as in mathematical analysis, to deduce usual formulas and the rules of differentiation.

As illustration let us consider

$$L(\xi, \beta) = \xi e^{\xi\beta} = \sum_{s=0}^{\infty} \frac{\xi^{s+1}\beta^s}{s!}. \tag{1.160}$$

Here

$$a_s(\xi) = \frac{\xi^{s+1}}{s!}, \quad a_s^{(m)}(\xi) = \frac{\xi^{s-m+1}}{(s-m)!} \cdot \frac{s+1}{s-(m-1)}_{s\to\infty} \to 0. \tag{1.161}$$

Conditions (1.150) are satisfied. Therefore

$$\frac{d}{d\xi}(\xi e^{\xi\beta}) = (1+\xi\beta)e^{\xi\beta} = \sum_{s=0}^{\infty}\frac{(s+1)\xi^s\beta^s}{s!},$$

$$\frac{d^2}{d\xi^2}(\xi e^{\xi\beta}) = \beta(2+\xi\beta)e^{\xi\beta} = \sum_{s=0}^{\infty}\frac{(s+1)\xi^{s-1}\beta^s}{(s-1)!} \text{ and the so forth} \qquad (1.162)$$

d) as in analysis, can be placed the question concerning the determination of original operator-function for this regular cperator-function L (ξ, β), i.e., concerning integration L (ξ, β):

$$\int L(\xi,\beta)\,d\xi = \int\left\{\sum_{s=0}^{\infty}a_s(\xi)\beta^s\right\}d\xi = \sum_{s=0}^{\infty}\left\{\int a_s(\xi)\,d\xi\right\}\beta^s. \qquad (1.163)$$

The last/latter expression is the abbreviated notation of the fcllowing (see (1.43))

$$\int L(\xi,\beta)\,d\xi\,\{\varphi(\eta)\} = \int\left[\sum_{s=0}^{\infty}a_s(\xi)\beta^s\right]d\xi\,\{\varphi(\eta)\} = \qquad (1.164)$$

$$= \sum_{s=0}^{\infty}\left[\int a_s(\xi)\,d\xi\right]\beta^s\{\varphi(\eta)\} = \sum_{s=0}^{\infty}\left[\int a_s(\xi)\,d\xi\right]\varphi^{(s)}(\eta) = \int\Phi(\xi,\eta)\,d\xi.$$

Integral (1.163) exists, since operator-function L (ξ,β) is continuous on ξ in interval/gap [0.1] (see (1.157)). The legitimacy cf the exchange of signs Σ and $\int$ in (1.163) and (1.164) and transition to integral of $\Phi(\xi,\eta)$ escape/ensues from the continuity of functions $a_s(\xi)$ and uniform relative to ξ convergence of series (1.143).

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METHOD OF INITIAL FUNCTIONS FOR TWO DIMENSIONAL BOUNDARY PROBLE--ETC(U)
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Further it is possible to introduce the concept of definite integral, integral with the alternating/variable upper limit, to justify usual rules and the formulas of integration. This is made just as in mathematical analysis for Riemann integrals. Therefore let us simply consider that the formulas of analysis are spread also to integrals of L (ξ, β) Let us note recently in the case of indefinite integral instead of the arbitrary constant one should write arbitrary operator $L_c(\beta)$. In formula (1.164) to this fact corresponds that fact that during the indefinite integration of function $\Phi(\xi,\eta)$ one should add arbitrary function $f(\eta)$.

One should also focus attention on certain special feature/peculiarity of indefinite integral of L (ξ, β). If is satisfied the condition

$$\overline{\lim_{s\to 0}} \sqrt[s]{\left|\int a_s(\xi)\,d\xi\right|} = 0, \qquad (1.165)$$

that a series in ((1.163) represents regular operator, i.e., under condition (1.165), by integrating the written in the form of a series operator-function L (ξ, β), again let us arrive at regular operator-function. However, integrating the closed expression L (ξ, β), with arbitrary $L_c(\beta)$ we can obtain irregular operator. For example, the operator

$$L(\xi, \beta) = \frac{\sin \xi (1-\beta)}{1-\beta} \qquad (1.166)$$

is regular, since function L (ξ, z) = $(\text{sine } \xi\ (1-z))/1-z$ whole (during supplementary definition L $(\xi, 1) = \xi$). By integrating it, let us find

$$L_2(\xi, \beta) = \int \frac{\sin \xi(1-\beta)}{1-\beta} d\xi = -\frac{\cos \xi(1-\beta)}{(1-\beta)^2} + L_c(\beta). \quad (1.167)$$

With $L_c(\beta) = L_c^+(\beta)$ operator L_2 (ξ, β) will be irregular, since function $L_2(\xi, z) = \frac{\cos \xi(1-z)}{(1-z)^2} + L_c^+(z)$ has a pole of the second order at point z = 1.

Operator, obtained as a result of the indefinite integration, always it is possible to make regular, after selecting properly $L_c(\beta)$. So, in (1.167) one should assume

$$L_c(\beta) = \frac{1}{(1-\beta)^2} + L_c^+(\beta). \quad (1.168)$$

Is simpler, however, to generally avoid similar phenomena.

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In order that the integration would not derive/conclude beyond limits many regular operator-functions, let us agree to apply only definite integrals with constant or changing limits (it is suggested to be carried out also condition (1.165)). For example,

$$\left.\begin{aligned}\int_0^1 \frac{\sin\xi(1-\beta)}{1-\beta}d\xi = \frac{1-\cos\xi(1-\beta)}{(1-\beta)^2},\\ \int_0^\xi \xi\sin\xi\beta d\xi = \frac{\sin\xi\beta-\xi\beta\cos\xi\beta}{\beta^2},\end{aligned}\right\} \quad (1.169)$$

Let us note still that the operators, similar that which stand in the right sides of equalities (1.166) and (1.169), thus far we consider as one-piece/entire symbols. Below (p. 6 pages 108) it will become clear, that with their defined stipulations it is possible to consider and as fractions.

e) because of the fact that are determined the actions of differentiation and integration, appears possibility to examine, differential equations for operator-functions. The methods of the integration of these equations and their substantiation will be in essence the same as for usual differential equations. For example, the integral of the equation

$$\frac{dL}{d\xi}+\beta L=\xi \quad (1.170)$$

under condition L $(0, \beta) = 0$ will be

$$L(\xi,\beta)=\frac{e^{-\xi\beta}-1+\xi\beta}{\beta^2}, \quad (1.171)$$

and the integral of the equation

$$\frac{d^2L}{d\xi^2}+L=0 \quad (1.172)$$

under the conditions L (0) = sh β and L' (0) = ch β will be

$$L(\xi, \beta) = \operatorname{sh}(\xi + \beta). \qquad (1.173)$$

Consequently, in order that the solution to equation would be the operator-function, but not simply function, into equation (first example) or under supplementary conditions (second example) as the parameter must enter the symbol β.

The solution to differential equation can render/show irregular operator. It is difficult to indicate the conditions, necessary and sufficient in order that the solution would be regular.

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So, the equation

$$\frac{d^2L}{d\xi^2} + \beta^2 L = e^{\xi\beta} \qquad (1.174)$$

has regular coefficients and right side; however, its general solution is represented by the operator

$$L(\xi, \beta) = A \sin \xi\beta + B \cos \xi\beta + \frac{e^{\xi\beta}}{2\beta^2} \qquad (1.175)$$

However there is no need under such conditions since in the future will be examined the irregular operators.

f) in conclusion let us do several observations.

1. Speaking about operator-function L (ξ, β), we we consider it possible to call it also either simply operator or simply by function (in the same way as this is made in the relation to vector function).

2. with any fixed/recorded $\xi = \xi_0$ L (ξ_0, β) it is converted into the usual regular operator whose properties were described and substantiated in p. 2 and of 3 present paragraphs. On the strength of uniform continuity L (ξ, β) and uniform (relatively ξ) convergence of series (1.143) all these properties are spread also to the regular operator-functions L (ξ, β). The number of permissible actions includes the differentiation with respect to symbol β (p. 3). Let us agree this action to designate by the symbcl of particular differentiation. So that, for example,

$$\frac{d}{d\xi}(e^{\xi\beta^2}) = (e^{\xi\beta^2})' = \beta^2 e^{\xi\beta^2}, \quad \frac{\partial}{\partial\beta}(e^{\xi\beta^2}) = 2\beta e^{\xi\beta^2}. \qquad (1.176)$$

For differentiation with respect to variable ξ let us apply both designation of Leibnitz and Lagrange's simpler designation (that also is done in (1.176)).

3. All reasonings were carried out for $\xi \in [0,1]$, since precisely

this interval/gap will be required by us subsequently. It is not difficult to see, however, that all results can be generalized, also, to wider intervals, for example (0, ∞). Actually this it is necessary to take into account only under conditions of form (1.150).

4. On the basis of observations p. 2 and 3g, let us find

$$\begin{gathered}\sin(\xi\beta \pm 2\pi) = \sin\xi\beta\cdot\cos 2\pi \pm \cos\xi\beta\cdot\sin 2\pi = \sin\xi\beta,\\ \cos(\xi\beta \pm 2\pi) = \cos\xi\beta\cdot\cos 2\pi \mp \sin\xi\beta\cdot\sin 2\pi = \cos\xi\beta,\\ \sin\left(\frac{\pi}{2} - \xi\beta\right) = \sin\frac{\pi}{2}\cos\xi\beta - \cos\frac{\pi}{2}\cdot\sin\xi\beta = \cos\xi\beta, \quad (1.177)\\ \sin(\pi - \xi\beta) = \sin\pi\cdot\cos\xi\beta - \cos\pi\cdot\sin\xi\beta = \sin\xi\beta,\\ \sin[k(\xi+\beta) \pm 2\pi] = \sin k(\xi+\beta)\end{gathered}$$

and so forth. This means that for the trigonometric operators are valid the "formula of bringing", and same these operators have the "period" of 2π.

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§6. Realization of the regular operators.

In the implementation (determinaticn of values) of the regular operators or operator-functions can be met the different questions: necessary to realize the assigned in the form of a series operator to that which was assigned in the form of a series function; necessary to realize arbitrary operator for the assigned function or, on the contrary, the assigned operator for arbitrary function; is known cperator's value L (ξ, β), it is required to find value L' (ξ, β) and sc forth.

All these questions will be solved, on the basis of formulas and the properties, found in §5. Let us assume those which were carried cut conditions (1.152). Then all the being encountered series will converge atsolutely and evenly, but therefore the application/use of formulas and properties §5 will be that which was substantiated. As a rule, in text we will not this specify.

1. Method.

Let

$$L(\xi, \beta)\varphi(\eta) = L^{+}(\xi, \beta)\varphi(\eta) = \Phi(\xi, \eta), \quad (1.178)$$

whereupon

$$L(\xi, \beta) = a_0(\xi) + a_1(\xi)\beta + a_2(\xi)\beta^2 + \ldots = \sum_{s=0}^{\infty} a_s(\xi)\beta^s, \quad (1.179)$$

$$\varphi(\eta) = c_0 + c_1\eta + c_2\eta^2 + \ldots = \sum_{n=0}^{\infty} c_n\eta^n, \quad (1.180)$$

$$\Phi(\xi, \eta) = A_0(\xi) + A_1(\xi)\eta + A_2(\xi)\eta^2 + \ldots = \sum_{p=0}^{\infty} A_p(\xi)\eta^p. \quad (1.181)$$

Cutside (1.179) - (1.181) in (1.178), we will obtain

$$\Phi(\xi, \eta) = \sum_{s=0}^{\infty}\sum_{n=0}^{\infty} a_s c_n \beta^s \eta^n = \sum_{p=0}^{\infty} A_p \eta^p, \quad (1.182)$$

where

$$\left.\begin{aligned} A_0 &= a_0c_0 + a_1c_1 + 2!a_2c_2 + 3!a_3c_3 + 4!a_4c_4 + \ldots, \\ A_1 &= a_0c_1 + 2!a_1c_2 + 3!a_2c_3 + 4!a_3c_4 + 5!a_4c_5 + \ldots, \\ A_2 &= \frac{1}{2!}(2!a_0c_2 + 3!a_1c_3 + 4!a_2c_4 + 5!a_3c_5 + 6!a_4c_6 + \ldots, \\ A_3 &= \frac{1}{3!}(3!a_0c_3 + 4!a_1c_4 + 5!a_2c_5 + 6!a_3c_6 + 7!a_4c_7 + \ldots, \end{aligned}\right\} \quad (1.183)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

i. e.,

$$A_p = \frac{1}{p!} \sum_{s=0}^{\infty} (s+p)!\, a_s c_{s+p}, \quad p = 0, 1, 2, \ldots \qquad (1.184)$$

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The obtained formulas make it possible to obtain the value of the assigned in the form series according to degrees β (or decomposed in this series) the regular operator above the assigned in the form of a series according to degrees (or decomposed in this series) function $\varphi(\eta)$. Operator's value also is obtained in the form of a series, its coefficients can be calculated, generally speaking, approximately, although to any degree of accuracy. In special cases series (1.183) and (1.81) can be convolute or they generally are

broken. Then $\Phi(\xi,\eta)$ is obtained in the closed form. Let us consider examples.

a) Let

$$L(\xi,\beta)=J_0(\xi,\beta)=\sum_{s=0}^{\infty}\frac{(-1)^s}{(s!)^2}\left(\frac{\xi}{2}\right)^{2s}\beta^{2s},$$

$$\varphi(\eta)=I_0(\eta)=\sum_{n=0}^{\infty}\frac{1}{(n!)^2\,2^{2n}}\,\eta^{2n}. \qquad (1.185)$$

Here

$$a_{2s}=\frac{(-1)^s}{(s!)^2}\,\frac{\xi^{2s}}{2^{2s}},\quad a_{2s+1}=0,\quad c_{2n}=\frac{1}{(n!)^2\,2^{2n}},\quad c_{2n+1}=0. \qquad (1.186)$$

According to (1.183)

$$\left.\begin{aligned} A_0 = 1 - \frac{2!\xi^2}{2^4} + \frac{4!\xi^4}{(2!)^4 2^8} - \ldots = 1 - 0{,}125\xi^2 + \\ + 0{,}00586\xi^4 - 0{,}000136\xi^6 + \ldots, \\ A_2 = \frac{1}{2!}\left\{\frac{2!}{2^2} - \frac{4!\xi^2}{(2!)^2 2^6} + \ldots\right\} = \\ = 0{,}25 - 0{,}0469\xi^2 + 0{,}00244\xi^4 - \ldots, \\ A_4 = \frac{1}{4!}\left\{\frac{4!}{(2!)^2 2^4} - \frac{6!\xi^2}{(3!)^2 2^8} + \ldots\right\} = \\ = 0{,}0156 - 0{,}00326\xi^2 + \ldots, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ A_{2p+1} = 0, \end{aligned}\right\} \quad (1.187)$$

and therefore

$$J_0(\xi, \beta) I_0(\eta) = 1 - 0{,}125\xi^2 + \ldots + (0{,}25 - 0{,}0469\xi^2 + \ldots)\eta^2 + + (0{,}0156 - 0{,}00326\xi^2 + \ldots)\eta^4 + \ldots . \quad (1.188)$$

b) let $L(\xi, \beta) = L^+(\xi, \beta)$ - arbitrary operator, and

$$\varphi(\eta) = e^{m\eta} = \sum_{n=0}^{\infty} \frac{m^n}{n!}\eta^n, \quad c_n = \frac{m^n}{n!}, \quad (1.189)$$

where

m - any complex number. From (1.184)

$$A_p = \frac{1}{p!}\sum_{s=0}^{\infty}(s+p)!\,a_s\frac{m^{s+p}}{(s+p)!} = \frac{m^p}{p!}\sum_{s=0}^{\infty}a_s m^s = \frac{m^p}{p!}L(\xi, m). \tag{1.190}$$

Further, from (1.81)

$$\Phi(\xi, \eta) = \sum_{p=0}^{\infty}\frac{m^p}{p!}L(\xi, m)\,\eta^p = L(\xi, m)\sum_{p=0}^{\infty}\frac{(m\eta)^p}{p!} = L(\xi, m)\cdot e^{m\eta}. \tag{1.191}$$

Consequently,

$$\boxed{L(\xi, \beta)\,e^{m\eta} = L(\xi, m)\,e^{m\eta}} \quad (m \text{ - any}). \tag{1.192}$$

Here L (ξ, m) - this is the already usual function, obtained from L (ξ, z) with z = m. Let us note incidentally that the used here method of obtaining this very important formula is not better/best.

The described method of realization, obviously, is completely universal (within the framework of the regular operators and $\varphi(\eta)\in\omega^{(A)}$), but in a number of cases it is excessively bulky.

2. Realization above polynomials, hyperbolic and trigonometric functions.

When the known closed expression for $\phi(\eta)$ and this expression sufficiently simple, it is expedient to use directly formula (1.143), i.e.,

$$L(\xi,\beta)\varphi(\eta)=L^{+}(\xi,\beta)\varphi(\eta)=\sum_{s=0}^{\infty}a_s(\xi)\varphi^{(s)}(\eta)=\Phi(\xi,\eta). \quad (1.193)$$

Let us consider concrete/specific/actual examples.

a) Let

$$\varphi(\eta)=\eta^n,\ n \text{ - is natural number.} \quad (1.194)$$

Then

$$\sum_{s=0}^{\infty}a_s\frac{d^s\eta^n}{d\eta^s}=\sum_{s=0}^{n}a_s(\xi)n(n-1)\ldots(n-s+1)\eta^{n-s}=n!\sum_{p=0}^{n}\frac{a_{n-p}(\xi)}{p!}\eta^p \quad (1.195)$$

and

$$L(\xi, \beta)\{\eta^n\} = n! \sum_{p=0}^{n} \frac{a_{n-p}(\xi)}{p!} \eta^p = n! \left[a_n(\xi) + a_{n-1}(\xi)\eta + a_{n-2}(\xi)\frac{\eta^2}{2!} + \right. \\ \left. + \ldots + a_0(\xi)\frac{\eta^n}{n!} \right]. \quad (1.196)$$

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So that, for example,

$$L(\xi, \beta)\{1 - 6\eta + 3\eta^2\} = a_0(\xi) - 6[a_1(\xi) + a_0(\xi)\eta] + \\ + 6\left[a_2(\xi) + a_1(\xi)\eta + \frac{1}{2}a_0(\xi)\eta^2\right]. \quad (1.197)$$

b) Let (we repeat the example b p. 1)

$$\varphi(\eta) = e^{m\eta}. \quad (1.198)$$

Then

$$L(\xi, \beta)e^{m\eta} = \sum_{s=0}^{\infty} a_s(\xi)m^s e^{m\eta} = e^{m\eta}\sum_{s=0}^{\infty} a_s(\xi)m^s = e^{m\eta}L(\xi, m). \quad (1.199)$$

which coincides with (1.192).

c) Function ϕ (η) is assigned/prescribed in the form absolutely

and evenly (with $\eta \in \Gamma$) of the converging series

$$\varphi(\eta) = \sum_{n=0}^{\infty} c_n e^{m_n \eta} \in \omega^{(A)}. \tag{1.200}$$

According to (1.192) we obtain

$$\Phi(\xi, \eta) = L(\xi, \beta)\varphi(\eta) = L(\xi, \beta)\sum_{n=0}^{\infty} c_n e^{m_n \eta} = \sum_{n=0}^{\infty} c_n L(\xi, m_n) e^{m_n \eta}. \tag{1.201}$$

The convergence of a last/latter series depends on the form of the target/purposes of function L (ξ, z) and must be specially investigated.

d) Let us consider action L (ξ, β) above hyperbolic functions. Taking into account (1.192), we find

$$L(\xi, \beta)\operatorname{ch} m\eta = L(\xi, \beta)\left\{\frac{e^{m\eta} + e^{-m\eta}}{2}\right\} = \frac{1}{2}[L(\xi, m)e^{m\eta} + \\ + L(\xi, -m)e^{-m\eta}], \tag{1.202}$$

cr

$$L(\xi, \beta)\operatorname{ch} m\eta = \frac{1}{2}[L(\xi, m) + L(\xi, -m)]\operatorname{ch} m\eta + \frac{1}{2}[L(\xi, m) - \\ - L(\xi, -m)]\operatorname{sh} m\eta \tag{1.203}$$

and, analogously,

$$L(\xi, \beta)\operatorname{sh} m\eta = \frac{1}{2}[L(\xi, m) + L(\xi, -m)]\operatorname{sh} m\eta + \frac{1}{2}[L(\xi, m) - \\ - L(\xi, -m)]\operatorname{ch} m\eta. \quad (1.204)$$

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Hence, in particular: if operator L (ξ, β) even relatively β, then

$$\boxed{L(\xi, \beta)\begin{matrix}\operatorname{ch}\\ \operatorname{sh}\end{matrix} m\eta = L(\xi, m)\begin{matrix}\operatorname{ch}\\ \operatorname{sh}\end{matrix} m\eta.} \quad (1.205)$$

if operator L (ξ, β) odd relatively β, then

$$\boxed{L(\xi, \beta)\begin{matrix}\operatorname{ch}\\ \operatorname{sh}\end{matrix} m\eta = L(\xi, m)\begin{matrix}\operatorname{sh}\\ \operatorname{ch}\end{matrix} m\eta.} \quad (1.206)$$

By substituting in (1.203) and (1.204) m on im, let us arrive at

$$L(\xi, \beta)\cos m\eta = \frac{1}{2}[L(\xi, im) + L(\xi, -im)]\cos m\eta + \\ + \frac{i}{2}[L(\xi, im) - L(\xi, -im)]\sin m\eta. \quad (1.207)$$

$$L(\xi, \beta)\sin m\eta = \frac{1}{2}[L(\xi, im) + L(\xi, -im)]\sin m\eta - \\ - \frac{i}{2}[L(\xi, im) - L(\xi, -im)]\cos m\eta.$$

Hence, in particular: if operator L (ξ, β) even relatively β, then

$$\boxed{L(\xi, \beta) \frac{\cos}{\sin} m\eta = L(\xi, im) \frac{\cos}{\sin} m\eta.} \qquad (1.208)$$

if operator L (ξ, β) odd relatively β, then

$$\boxed{L(\xi, \beta) \frac{\cos}{\sin} m\eta = \pm iL(\xi, im) \frac{\sin}{\cos} m\eta.} \qquad (1.209)$$

So that, for example,

$$\beta^2 \sin\beta \{\sin m\eta\} = -m^2 \operatorname{sh} \cos m\eta. \qquad (1.210)$$

3. Application/use of the operators to the product of functions.

a) Let us use (1.193) to function $\phi(\eta) = u(\eta)\, v(\eta)$

$$L(\xi, \beta)\{u(\eta)\, v(\eta)\} = \sum_{s=0}^{\infty} a_s(\xi)\, \beta^s \{u(\eta)\, v(\eta)\} \qquad (1.211)$$

and introduce in operator β indices u and v, which indicate, to which function is spread its acticn.

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Thus, for instance,

$$\left.\begin{array}{l} \beta_u u(\eta) = \beta u(\eta) = u'(\eta), \quad \beta_v v(\eta) - \beta v(\eta) = v'(\eta), \\ \beta_u^s u(\eta) = u^{(s)}(\eta), \quad \beta_v^s v(\eta) = v^{(s)}(\eta), \\ \beta_u^s \{uv\} = u^{(s)}(\eta)\, v(\eta), \quad \beta_u^l \beta_v^s \{uv\} = u^{(l)}(\eta)\, v^{(s)}(\eta), \\ \beta_u^s v(\eta) \text{ and } \beta_v^s u(\eta) \text{ do not have sense.} \end{array}\right\} \qquad (1.212)$$

Then

$$\beta^s \{uv\} = (\beta_u + \beta_v)^s \{uv\} = \sum_{l=0}^{s} C_s^l \beta_u^l u(\eta) \cdot \beta_v^{s-l} v(\eta) = \sum_{l=0}^{s} C_s^l u^{(l)}(\eta)\, v^{(s-l)}(\eta) \qquad (1.213)$$

and

$$L(\xi,\beta)\{u(\eta)v(\eta)\} = \sum_{s=0}^{\infty} a_s(\xi)\left[\sum_{j=0}^{s} C_s^j u^{(j)}(\eta) v^{(s-j)}(\eta)\right] =$$
$$= a_0 uv + a_1(u'v + uv') + a_2(u''v + 2u'v' + uv'') + \cdots. \quad (1.214)$$

b) Let, for example,

$$\varphi(\eta) = \eta^n e^{m\eta}, \quad u = \eta^n, \quad v = e^{m\eta}, \quad (1.215)$$

where n is natural number. From (1.214) we will obtain

$$L(\xi,\beta)\{\eta^n e^{m\eta}\} = \sum_{s=0}^{\infty} a_s\left[\sum_{j=0}^{s} C_s^j n(n-1)\cdots(n-j+1)m^{s-j}\eta^{n-j}e^{m\eta}\right] =$$
$$= e^{m\eta}\sum_{j=0}^{n} C_n^j\left[\sum_{s=0}^{\infty} a_s\cdot s(s-1)\cdots(s-j+1)m^{s-j}\right]\eta^{n-j} =$$
$$= e^{m\eta}\sum_{j=0}^{n} C_n^j \frac{\partial^j L(\xi,m)}{\partial m^j}\eta^{n-j} = e^{m\eta}\sum_{j=0}^{n} C_n^j \frac{d^j}{dm^j}\eta^{n-j}L(\xi,m) =$$
$$= e^{m\eta}\left(\eta + \frac{\partial}{\partial m}\right)^n L(\xi,m). \quad (1.216)$$

Thus,

$$\boxed{L(\xi, \beta)\{\eta^n e^{mn}\} = e^{mn}\left(\eta + \frac{\partial}{\partial m}\right)^n L(\xi, m).} \qquad (1.217)$$

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By substituting here m - on - m and by deducting result from (1.217), let us find

$$L(\xi,\beta)\left\{\eta^n \begin{matrix}\mathrm{ch}\\ \mathrm{sh}\end{matrix}\, m\eta\right\} = \frac{1}{2}\begin{matrix}\mathrm{ch}\\ \mathrm{sh}\end{matrix}\, m\eta\left[\left(\eta+\frac{\partial}{\partial m}\right)^n L(\xi,m)+\left(\eta-\frac{\partial}{\partial m}\right)^n + \right.$$
$$\left. + L(\xi,-m)\right] + \frac{1}{2}\begin{matrix}\mathrm{sh}\\ \mathrm{ch}\end{matrix}\, m\eta\left[\left(\eta+\frac{\partial}{\partial m}\right)^n L(\xi,m)-\left(\eta-\frac{\partial}{\partial m}\right)^n L(\xi,-m)\right]. \tag{1.218}$$

In particular with n = 1 let us have

$$L(\xi,\beta)\left\{\eta \begin{matrix}\mathrm{ch}\\ \mathrm{sh}\end{matrix}\, m\eta\right\} = \frac{1}{2}\begin{matrix}\mathrm{ch}\\ \mathrm{sh}\end{matrix}\, m\eta\cdot\left\{\eta\left[L(\xi,m)+L(\xi,-m)\right] + \right.$$
$$\left. + \frac{\partial}{\partial m}\left[L(\xi,m)-L(\xi,-m)\right]\right\} + \frac{1}{2}\begin{matrix}\mathrm{sh}\\ \mathrm{ch}\end{matrix}\, m\eta\left\{\eta\left[L(\xi,m)-L(\xi,-m)\right] + \right.$$
$$\left. + \frac{\partial}{\partial m}\left[L(\xi,m)+L(\xi,-m)\right]\right\}, \tag{1.219}$$

whence with the even relatively β operator

$$\boxed{L(\xi,\beta)\left\{\eta \begin{matrix}\mathrm{ch}\\ \mathrm{sh}\end{matrix}\, m\eta\right\} = L(\xi,m)\eta \begin{matrix}\mathrm{ch}\\ \mathrm{sh}\end{matrix}\, m\eta + \frac{\partial L(\xi,m)}{\partial m}\begin{matrix}\mathrm{sh}\\ \mathrm{ch}\end{matrix}\, m\eta.} \tag{1.220}$$

and with the odd relatively β operator

$$\boxed{L(\xi,\beta)\left\{\eta \begin{matrix}\mathrm{ch}\\ \mathrm{sh}\end{matrix}\, m\eta\right\} = L(\xi,m)\eta \begin{matrix}\mathrm{sh}\\ \mathrm{ch}\end{matrix}\, m\eta + \frac{\partial L(\xi,m)}{\partial m}\begin{matrix}\mathrm{ch}\\ \mathrm{sh}\end{matrix}\, m\eta.} \tag{1.221}$$

So that, for example,

$$\cos\beta\{\eta\,\mathrm{ch}\,m\eta\} = \cos m\cdot\eta\,\mathrm{ch}\,m\eta - \sin m\cdot\mathrm{sh}\,m\eta. \tag{1.222}$$

After replacing in (1.219) - (1.221) m on im (as this was done in p. 2d), let us find analogous expressions for $\varphi(\eta)=\eta\,{}^{\cos}_{\sin}\, m\eta.$

4. Operator L (ξ, β + k).

a) let us assume in (1.214) and $u=e^{k\eta}$ (k - any), v = ϕ (η). This will give

$$L(\xi,\beta)\{e^{k\eta}\varphi(\eta)\}=\sum_{s=0}^{\infty}a_s(\xi)\left[\sum_{j=0}^{s}C_s^j k^j e^{k\eta}\beta^{s-j}\,\varphi(\eta)\right]=$$
$$=e^{k\eta}\left(\sum_{s=0}^{\infty}a_s(\xi)\left[\sum_{j=0}^{s}C_s^j k^j\beta^{s-j}\right]\right)\{\varphi(\eta)\}=e^{k\eta}\left(\sum_{s=0}^{\infty}a_s(\xi)(k+\beta)^s\right)\{\varphi(\eta)\}, \tag{1.223}$$

i. e.,

$$L(\xi,\beta)\{e^{k\eta}\varphi(\eta)\}=e^{k\eta}L(\xi,\beta+k)\{\varphi(\eta)\}. \tag{1.224}$$

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Hence

$$\boxed{L(\xi, \beta + k)\,\varphi(\eta) = e^{-k\eta} L(\xi, \beta)\,[e^{k\eta}\varphi(\eta)]}. \qquad (1.225)$$

which, actually, coincides with formula (1.85) for the polynomial operators.

The obtained formula makes it possible to find operator's value L (ξ, β + k), if is known operator's value L (ξ, β).

b) Let us assume $\varphi(\eta) = e^{m\eta}$. Then, taking into account (1.192), from (1.225) we find

$$\begin{aligned} L(\xi, \beta + k)\,e^{m\eta} &= e^{-k\eta} L(\xi, \beta)\,\{e^{(m+k)\eta}\} = \\ &= e^{-k\eta} L(\xi, m + k)\,e^{(m+k)\eta} = L(\xi, m + k)\,e^{m\eta}. \end{aligned} \qquad (1.226)$$

c) After assuming in (1.225) $\varphi(\eta) = \eta^n$ and after taking into consideration (1.217), we will obtain

$$\begin{aligned} L(\xi, \beta + k)\,\{\eta^n\} &= e^{-k\eta} L(\xi, \beta)\,\{\eta^n e^{k\eta}\} = \\ &= e^{-k\eta} e^{k\eta}\left(\eta + \frac{\partial}{\partial k}\right)^n L(\xi, k) = \left(\eta + \frac{\partial}{\partial k}\right)^n L(\xi, k). \end{aligned} \qquad (1.227)$$

whence, in particular, with n = 1:

$$L(\xi, \beta + k)\,\{\eta\} = \eta L(\xi, k) + \frac{\partial L(\xi, k)}{\partial k}. \qquad (1.228)$$

and with n = 0

$$L(\xi, \beta + k)\{1\} = L(\xi, k). \qquad (1.229)$$

5. Operator $\partial L(\xi, \beta)/\partial \beta$

Let us assume in (1.214) and u (η) = η, v (η) = ϕ (η). Then

$$L(\xi, \beta)\{\eta\varphi(\eta)\} = \sum_{s=0}^{\infty} a_s(\xi)\left[\sum_{l=0}^{s} C_s^l \eta^{(l)} \beta^{s-l} \varphi(\eta)\right] =$$
$$= \sum_{s=0}^{\infty} a_0(\xi)[C_s^0 \eta\beta^s \varphi(\eta) + C_s^1 \beta^{s-1} \varphi(\eta)] = \sum_{s=0}^{\infty} a_0(\xi)[\eta\beta^s\varphi(\eta) + s\beta^{s-1}\varphi(\eta)] =$$
$$= \eta \sum_{s=0}^{\infty} a_0(\xi)\beta^s\varphi(\eta) + \sum_{s=0}^{\infty} a_0(\xi)[\eta\beta^s\varphi(\eta) + s\beta^{s-1}\varphi(\eta)], \quad (1.230)$$

or, taking into account (1.134),

$$L(\xi, \beta)\{\eta\varphi(\eta)\} = \eta L(\xi, \beta)\varphi(\eta) + \frac{\partial L(\xi, \beta)}{\partial \beta}\{\varphi(\eta)\}, \qquad (1.231)$$

whence

$$\boxed{\frac{\partial L(\xi, \beta)}{\partial \beta}\{\varphi(\eta)\} = L(\xi, \beta)\{\eta\varphi(\eta)\} - \eta L(\xi, \beta)\{\varphi(\eta)\}.} \qquad (1.232)$$

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This formula makes it possible to find operator's value $\frac{\partial L(\xi, \beta)}{\partial \beta}$, if is known operator's value L (ξ, β).

By examining $L(\xi, \beta)\{\eta^n \varphi(\eta)\}$ or by applying repeatedly several times (1.232), let us arrive at the following formulas:

$$\left.\begin{array}{r} \frac{\partial^2 L(\xi, \beta)}{\partial \beta^2}\{\varphi(\eta)\} = L(\xi, \beta)\{\eta^2 \varphi(\eta)\} - 2\eta L(\xi, \beta)\{\eta \varphi(\eta)\} + \\ + \eta^2 L(\xi, \beta)\{\varphi(\eta)\}, \\ \frac{\partial^3 L(\xi, \beta)}{\partial \beta^3}\{\varphi(\eta)\} = L(\xi, \beta)\{\eta^3 \varphi(\eta)\} - 3\eta L(\xi, \beta)\{\eta^2 \varphi(\eta)\} + \\ + 3\eta^2 L(\xi, \beta)\{\eta \varphi(\eta)\} - \eta^3 L(\xi, \beta)\{\varphi(\eta)\} \end{array}\right\} \quad (1.233)$$

and so forth.

6. Exponential, hyperbolic and trigonometric operators.

Let us consider now the concrete/specific/actual operator

$$L(k, \xi, \beta) = e^{k\xi\beta} = \sum_{s=0}^{\infty} \frac{(k\xi)^s}{s!} \beta^s, \quad (1.234)$$

where k - any real or complex number. Here

$$a_s = \frac{(k\xi)^s}{s!}, \quad a_s^{(m)} = \frac{(k\xi)^{s-m}}{(s-m)!}, \tag{1.235}$$

and ~~A~~. which means, are satisfied conditions (1.150) (with $|k| < R < \infty$).

Therefore

$$e^{k\xi\beta}\varphi(\eta) = \sum_{s=0}^{\infty}\left[\frac{(k\xi)^s}{s!}\beta^s\varphi(\eta)\right] = \sum_{s=0}^{\infty}\frac{(k\xi)^s}{s!}\varphi^{(s)}(\eta),$$

i. e.

$$\boxed{e^{k\xi\beta}\varphi(\eta) = \varphi(\eta + k\xi) = \Phi(\xi, \eta),} \tag{1.236}$$

whereupon $\Phi \in \omega^{(A)}$, if only are implemented and condition (1.152).

Substituting here k on - k, we find

$$\boxed{\begin{matrix}\mathrm{sh}\\ \mathrm{ch}\end{matrix} k\xi\beta\varphi(\eta) = \frac{e^{k\xi\beta} \mp e^{-k\xi\beta}}{2}\varphi(\eta) - \frac{\varphi(\eta + k\xi) \mp \varphi(\eta - k\xi)}{2} = \Phi(\xi, \eta).} \tag{1.237}$$

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Finally, after replacing here k by ik, we will obtain

$$\boxed{\begin{aligned}\sin k\xi\beta\varphi(\eta) &= \frac{\varphi(\eta + ik\xi) - \varphi(\eta - ik\xi)}{2i} = \mathrm{Im}\,\{\varphi(\eta + ik\xi)\} = \Phi(\xi, \eta).\\ \cos k\xi\beta\varphi(\eta) &- \frac{\varphi(\eta + ik\xi) + \varphi(\eta - ik\xi)}{2} = \mathrm{Re}\,\{\varphi(\eta + ik\xi)\} = \Phi(\xi, \eta).\end{aligned}} \tag{1.238}$$

It is not difficult to see that under condition (1.152) for all formulas (1.237) - (1.238) will be $\Phi \in \omega^{(A)}$.

The obtained formulas are completely analogous to formulas (1.64), (1.76) and (1.78), which were found at considerably more common/general/total assumptions relative to function ϕ ( ). In exactly the same manner it is possible to find the realization of the giperbolo-trigonometric operators, who correspond to the functions of acad. A. N. Krylov

$$y_1(k, \xi, \beta) = \operatorname{ch} k\xi\beta \cos k\xi\beta\varphi(\eta) = \frac{1}{4}\{\varphi[\eta + (1+i)k\xi] + \varphi[\eta - (1+i)k\xi] + \varphi[\eta + (1-i)k\xi] + \varphi[\eta - (1-i)k\xi]\} \quad (1.239)$$

and of so forth (see (1.8.1)).

7. Examples of realization.

Let us give several examples, which illustrate the application/use of the obtained in the present paragraph common/general/total formulas.

a) Utilizing permutability of the regular operators (page 44) and of formulas (1.237), (1.238), we find

$$\left.\begin{aligned}&\beta^n \sin k\xi\beta\varphi(\eta) = \sin k\xi\beta\cdot\beta^n\varphi(\eta) = \operatorname{Im}\{\varphi^{(n)}(\eta+ik)\},\\&\beta^n \cos k\xi\beta\varphi(\eta) = \cos k\xi\beta\cdot\beta^n\varphi(\eta) = \operatorname{Re}\{\varphi^{(n)}(\eta+ik)\},\\&\beta^n {\operatorname{ch}\atop\operatorname{sh}} k\xi\beta\varphi(\eta) = {\operatorname{ch}\atop\operatorname{sh}} k\xi\beta\cdot\beta^n\varphi(\eta) = \frac{\varphi^{(n)}(\eta+k\xi)\pm\varphi^{(n)}(\eta+k\xi)}{2}.\end{aligned}\right\} \quad (1.240).$$

b) from (1.205) and (1.206), (1.208) and (1.209), (1.237) and (1.238) let us have

$$\left.\begin{aligned}&\operatorname{sh}k\xi\beta\begin{Bmatrix}\operatorname{sh}\\ \operatorname{ch}\end{Bmatrix}m\eta = \operatorname{sh}k\xi m\cdot{\operatorname{ch}\atop\operatorname{sh}}m\eta,\\&\operatorname{sh}k\xi\beta\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}m\eta = \begin{cases}\sin k\xi m\cdot\cos m\eta,\\-\sin k\xi m\cdot\sin m\eta,\end{cases}\\&\sin k\xi\beta\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}m\eta = \begin{cases}\operatorname{sh}k\xi m\cdot\cos m\eta,\\-\operatorname{sh}k\xi m\cdot\sin m\eta\end{cases}\end{aligned}\right\} \quad (1.241).$$

and so forth.

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c) From (1.238) we find

$$\left.\begin{aligned}&\sin k\xi\beta\{1\} = \mathrm{Im}\{1\} = 0, \quad \cos k\xi\beta\{1\} = \mathrm{Re}\{1\} = 1,\\&\sin k\xi\beta\{\eta\} = \mathrm{Im}\{\eta + ik\xi\} = k\xi,\\&\qquad \cos k\xi\beta\{\eta\} = Re\{\eta + ik\xi\} = \eta,\\&\qquad \sin k\xi\beta\{\eta^2\} = \mathrm{Im}\{(\eta + ik\xi)^2\} = 2k\xi\eta\end{aligned}\right\} \quad (1.242)$$

and generally with n natural

$$\left.\begin{aligned}
&\sin k\xi\beta\{\eta^n\} = nk\xi\eta^{n-1} - \frac{n(n-1)(n-2)}{3!}k^3\xi^3\eta^{n-3} + \\
&+ \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}k^5\xi^5\eta^{n-5} - \ldots + \\
&+ \begin{cases} (-1)^{\frac{n-2}{2}}(k\xi)^{n-1}\eta \text{ when } n = 2s, \\ (-1)^{\frac{n-1}{2}}(k\xi)^n \text{ when } n = 2s+1, \end{cases} \\
&\cos k\xi\beta\{\eta^n\} = \eta^n - \frac{n(n-1)}{2!}k^2\xi^2\eta^{n-2} + \\
&+ \frac{n(n-1)(n-2)(n-3)}{4!}k^4\xi^4\eta^{n-4} - \ldots + \\
&+ \begin{cases} (-1)^{\frac{n}{2}}(k\xi)^n \text{ when } n = 2s, \\ (-1)^{\frac{n-1}{2}}(k\xi)^{n-1}\eta \text{ when } n = 2s+1. \end{cases}
\end{aligned}\right\} \quad (1.243)$$

d) By using the property of operator-functions, expressed by formula (1.164), let us integrate over ξ the second of equalities

(1.237)

$$\int_0^{\xi} \operatorname{ch} k\xi\beta\varphi(\eta)\,d\xi = \left(\int_0^{\xi} \operatorname{ch} k\xi\beta d\xi\right)\left\{\varphi(\eta)\right\} = \frac{\operatorname{sh} k\xi\beta}{k\beta}\varphi(\eta) =$$

$$= \frac{1}{2}\int_0^{\xi}\varphi(\eta + k\xi)\,d\xi + \frac{1}{2}\int_0^{\xi}\varphi(\eta - k\xi)\,d\xi = \frac{1}{2k}\int_0^{\eta+k\xi}\varphi(\zeta)\,d\zeta -$$

$$- \frac{1}{2k}\int_0^{\eta-k\xi}\varphi(\zeta)\,d\zeta = \frac{1}{2k}\int_{\eta-k\xi}^{\eta+k\xi}\varphi(\zeta)\,d\zeta. \tag{1.244}$$

i. e.,

$$\boxed{\frac{\operatorname{sh} k\xi\beta}{\beta}\varphi(\eta) = \left(k\xi + \frac{k^3\xi^3}{3!}\beta^2 + \cdots\right)\varphi(\eta) = \frac{1}{2}\int_{\eta-k\xi}^{\eta+k\xi}\varphi(\zeta)\,d\zeta.} \tag{1.245}$$

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Let us replace k for ik

$$\boxed{\frac{\sin k\xi\beta}{\beta}\varphi(\eta) = \left(k\xi - \frac{k^3\xi^3}{3!}\beta^2 + \cdots\right)\varphi(\eta) = \frac{1}{2i}\int_{\eta-ik\xi}^{\eta+ik\xi}\varphi(\zeta)\,d\zeta.} \tag{1.246}$$

e) By using (1.245) and (1.225), let us find

$$\frac{\operatorname{sh} m\xi(\beta+k)}{\beta+k}\varphi(\eta)=\left(\frac{\operatorname{sh} m\xi k}{k}+\frac{m\xi k\operatorname{ch} m\xi k-\operatorname{sh} m\xi k}{k^2}\beta+\cdots\right)\varphi(\eta)=$$
$$=\frac{1}{2}\int\limits_{\eta-m\xi}^{\eta+m\xi} e^{k(\zeta-\eta)}\varphi(\zeta)\,d\zeta. \qquad (1.247)$$

Then

$$\frac{\sin m\xi(\beta+k)}{\beta+k}=\frac{1}{2i}\int\limits_{\eta-im\xi}^{\eta+im\xi} e^{k(\zeta-\eta)}\varphi(\zeta)\,d\zeta. \qquad (1.248)$$

Let us note that here m and k - any.

f) Let us assume in (1.232) $L=\frac{\operatorname{sh} k\xi\beta}{\beta}$. Taking into account (1.245) this it will give

$$\left(\frac{\partial}{\partial\beta}-\frac{\operatorname{sh}k\xi\beta}{\beta}\right)\varphi(\eta)=\frac{k\xi\beta\operatorname{ch}k\xi\beta-\operatorname{sh}k\xi\beta}{\beta^2}\varphi(\eta)=\frac{\operatorname{sh}k\xi\beta}{\beta}\{\eta\varphi(\eta)\}-$$

$$-\eta\frac{\operatorname{sh}k\xi\beta}{\beta}\{\varphi\}=\frac{1}{2}\int\limits_{\eta-k\xi}^{\eta+k\xi}\zeta\varphi(\zeta)\,d\zeta-\frac{1}{2}\eta\int\limits_{\eta-k\xi}^{\eta+k\xi}\varphi(\zeta)\,d\zeta, \quad (1.249)$$

whence

$$\frac{k\xi\beta\operatorname{ch}k\xi\beta-\operatorname{sh}k\xi\beta}{\beta^2}\varphi(\eta)=\left(\frac{2}{3!}k^3\xi^3\beta-\frac{4}{5!}k^5\xi^5\beta^3+\cdots\right)\varphi(\eta)=$$

$$=\frac{1}{2}\int\limits_{\eta-k\xi}^{\eta+k\xi}(\zeta-\eta)\,\varphi(\zeta)\,d\zeta. \quad (k - \text{любое}) \quad (1.250)$$

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8. The equations, by which satisfy the values of the operators.

operator's value Φ - this is function of one is $\Phi(\eta)$ - either

two - $\Phi(\xi,\eta)$ - variables, according to that, will be L operator L (β) or operator-function L (ξ, β). In certain cases (this depends on form L) it is possible without the special work to establish/install, which differential equation satisfies function $\Phi(\xi,\eta)\in\omega^{(A)}$ regardless of the fact, above which function $\varphi\in\omega^{(A)}$ will act operator L. Let us give examples.

a) Let

$$L\varphi = \sin k\xi\beta\varphi = \Phi(\xi, \eta). \tag{1.251}$$

Twice we differentiate (in sense (1.151)) this identity with respect to ξ:

$$\frac{d^2L}{d\xi^2}\varphi = -k^2\beta^2 \sin k\xi\beta\varphi = \frac{\partial^2\Phi}{\partial\xi^2}, \tag{1.252}$$

and then twice on η:

$$\beta^2 L\varphi = \beta^2 \sin k\xi\beta\varphi = \frac{\partial^2\Phi}{\partial\eta^2}. \tag{1.253}$$

After multiplying the latter on k^2 and after forming with (1.252), we will obtain

$$\frac{\partial^2\Phi}{\partial\xi^2} + k^2\frac{\partial^2\Phi}{\partial\eta^2} = 0. \tag{1.254}$$

b) Let

$$L\varphi = e^{\xi\beta^2}\varphi = \Phi(\xi, \eta). \tag{1.255}$$

It differentiated this identity on ξ, and then twice on η and after subtracting results, let us find

$$\frac{\partial\Phi}{\partial\xi} - \frac{\partial^2\Phi}{\partial\eta^2} = 0. \qquad (1.256)$$

§7. Singular operators.

In the subsequent chapters for us it is necessary to deal in essence with the regular operators. Therefore and in the present chapter to the singular operators is given considerably less attention than regular.

1. Determination of the singular and mixed operator.

a) Operator L (β) let us call singular, if the expansion in a series of function $L(z) \longleftrightarrow L(\beta)$

in the vicinity of the infinitely receded point contains the cnly negative degrees of z:

$$L(\beta) \longleftrightarrow L(z) = \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots = \sum_{s=1}^{\infty} a_{-s} z^{-s}. \qquad (1.257)$$

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If necessary to emphasize operator's singularity L (β) let us him designate $L^-(\beta)$. The general view of singular operator it will be

$$L(z) \longleftrightarrow L(\beta) = \frac{a_{-1}}{\beta} + \frac{a_{-2}}{\beta^2} + \cdots = \sum_{s=1}^{\infty} a_{-s}\beta^{-s}. \quad (1.258)$$

If the coefficients a_s are the functions of variable $\xi \in [0, 1]$, then

$$L(\xi, z) \longleftrightarrow L(\xi, \beta) = \sum_{s=1}^{\infty} a_{-s}(\xi)\beta^{-s} \quad (1.259)$$

it is called singular operator-function.

By applying $L^-(\beta)$ either $L^-(\xi, \beta)$ to $\varphi(\eta)(\in \omega_L-$, we will obtain singular operator's value

$$L^-(\beta)\{\varphi(\eta)\} = \sum_{s=1}^{\infty} a_{-s}(\beta^{-s}\{\varphi(\eta)\}) = \sum_{s=1}^{\infty} a_{-s}\beta^{-s}\varphi = \Phi(\eta) \quad (1.260)$$

or

$$L^-(\xi, \beta)\{\varphi(\eta)\} = \sum_{s=1}^{\infty} a_{-s}(\xi)(\beta^{-s}\{\varphi(\eta)\}) = \sum_{s=1}^{\infty} a_{-s}(\xi)\beta^{-s}\varphi = \Phi(\xi, \eta). \quad (1.261)$$

The existence domain of operator ω_L- is determined from the condition cf the convergence of the entering in (1.260) and (1.261) series.

If the Loran expansion $L(z) \leftrightarrow L(\beta)$

in the vicinity of the infinitely receded point takes the form

$$L(\beta) \longleftrightarrow L(z) = \sum_{s=-\infty}^{\infty} a_s z^s, \qquad (1.262)$$

where at least one a_s, $s > 0$, excellently from zero, then the operator

$$L(z) \longleftrightarrow L(\beta) = \sum_{s=-\infty}^{\infty} a_s \beta^s \qquad (1.263)$$

let us name that which was mixed.

b) the introduced operators cannot have a sense, while not the rating value of symbol $\frac{1}{\beta^s} = \beta^{-s}$. In operational calculus it is accepted differential operator to determine not according to (1.2) and (1.24), but according to the formulas

$$\beta\varphi = \frac{d\varphi}{d\eta} + \varphi(0),$$

$$\beta^s\varphi = \varphi^{(s)}(\eta) + \varphi^{(s-1)}(0) + \varphi^{(s-2)}(0)\beta + \cdots + \varphi(0)\beta^{s-1}$$
$$(s = 1, 2, 3, \ldots). \qquad (1.264)$$

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Inverse operators (integral operators)

$$\frac{1}{\beta}\varphi = \beta^{-1}\varphi = \int_0^{\eta} \varphi(\zeta)\, d\zeta,$$

$$\frac{1}{\beta^s}\varphi = \beta^{-s}\varphi = \frac{1}{(s-1)!}\int_0^{\eta} (\eta-\zeta)^{s-1}\varphi(\zeta)\, d\zeta \quad (s = 1, 2, 3, \ldots) \quad (1.265)$$

are obtained interchangeable with the operators $\beta^n (n \gtreqless s)$. This fact turns out to be very valuable during the practical application/use of operational calculus, which deals almost exclusively with the singular (according to the taken here terminology) operators.

We, however, forewent definitions (1.264), since they extremely complicate the realization of the regular operators, with whom, as already mentioned, in essence it is necessary to be encountered subsequently, and they stopped at determinations (1.2) and (1.24). As concerns the operators $\frac{1}{\beta^s}$, let us take for them as in operational calculus, determination (1.265). In this case let us consider that η - the real or complex variable, which passes is certain finite domain Γ (page 39), at least, for example, (1.1). But then integral operators of differentiation and become noncommutative:

$$\beta^n \cdot \frac{1}{\beta^s} \neq \frac{1}{\beta^s} \cdot \beta^n. \qquad (1.266)$$

For example,

$$\beta^5 \cdot \frac{1}{\beta^3} \varphi(\eta) = \varphi''(\eta), \qquad (1.267)$$

and

$$\frac{1}{\beta^3} \beta^5 \varphi(\eta) = \frac{1}{2} \int_0^\eta (\eta - \zeta)^2 \varphi^{(5)}(\zeta)\, d\zeta = \varphi''(\eta) - \varphi''(0) - \varphi^{(3)}(0)\, \eta - \varphi^{(4)}(0) \frac{\eta^2}{2}. \qquad (1.268)$$

This, it is logical, it requires precaution and attention in inversion with the combinations of the regular and singular operators. The noted inconvenience do not cause, in our opinion,

great difficulties (as, for example, and the noncommutativity of matrix/dies does not impede their wide propagation).

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But if in inversion are located the only singular operators, then generally similar difficulties do not appear, since of (1.265) easily is perceived the interchangeability of integral operators:

$$\frac{1}{\beta^s}\cdot\frac{1}{\beta^n}=\beta^{-s}\cdot\beta^{-n}=\frac{1}{\beta^n}\cdot\frac{1}{\beta^s}=\beta^{-n}\cdot\beta^{-s}=\beta^{-(n+s)}. \qquad (1.269)$$

It follows from there that

$$\beta^s\cdot\frac{1}{\beta^n}=\beta^s\cdot\beta^{-n}=\beta^{s-n}, \quad (s\geqslant n\geqslant 0), \qquad (1.270)$$

that means in particular

$$\beta^s\cdot\frac{1}{\beta^s}=\beta^s\cdot\beta^{-s}=1, \qquad (1.271)$$

i. e.,

$$\beta^s=(\beta^{-s})^{-1} \qquad (1.272)$$

- the differential operator is reverse/inverse with respect to integral operator. Differential operator during determinations (1.2), (1.24) and (1.265) does not have inverse operator. It is possible to demonstrate the more common/general/total affirmation: it is not possible to construct the operators, reverse/inverse for (1.2) and

(1.24).

True, it is possible tc attain partial interchangeability β^s and $\frac{1}{\beta^n}$, after introducing the determination

$$\frac{1}{\beta^s}\varphi=\frac{1}{(s-1)!}\int\limits_0^{\eta}(\eta-\zeta)^{s-1}\varphi(\zeta)\,d\zeta+\sum_{p=0}^{s-1}\frac{\varphi^{(p)}(0)}{p!}\eta^p. \quad (1.273)$$

In this case $\beta^{-s}\cdot\beta^s=\beta^s\cdot\beta^{-s}=1,$ but will be preserved as before inequality (1.266) with n $\neq$ s.

c) on the basis of (1.260) and (1.265), we obtain

$$L^-(\beta)\varphi(\eta)=\sum_{s=1}^{\infty}a_{-s}\beta^{-s}\varphi=\sum_{s=1}^{\infty}\frac{a_{-s}}{(s-1)!}\int\limits_0^{\eta}(\eta-\zeta)^{s-1}\varphi(\zeta)\,d\zeta=\Phi(\eta) \quad (1.274)$$

cr, on the strength of the interchangeability of fold,

$$L^-(\beta)\varphi(\eta)=\sum_{s=1}^{\infty}a_{-s}\beta^{-s}\varphi=\sum_{s=1}^{\infty}\frac{a_{-s}}{(s-1)!}\int\limits_0^{\eta}\zeta^{s-1}\varphi(\eta-\zeta)\,d\zeta=\Phi(\eta). \quad (1.275)$$

Analogous expressions are obtained from (1.261) for operator-function $L^-(\xi,\beta)$.

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Let us explain the conditions, sufficient in order that entering in (1.274) (but, and also, therefore, (1.275)) a series would converge absolutely and evenly.

According to Weierstrass's sign/criterion, if

$$\left|\frac{a_{-s}}{(s-1)!}\int_0^{\eta}(\eta-\zeta)^{s-1}\varphi(\zeta)\,d\zeta\right|<\frac{1}{s^{\sigma}},\quad \sigma>1, \tag{1.276}$$

that series (1.274) converges absolutely and evenly, since converges a series

$$\sum_{s=1}^{\infty}\frac{1}{s^{\sigma}}.$$

Let us designate by R distance from point z = 0 to the most receded from the origin of coordinates singular point of function L (z) (according to (1.257) all the singular points of this function they lie/rest on the final distance), also, through M maximum $|L\ (z)|$ on circumference $|z| = \rho > R$. Then, applying the estimations of the module/moduli of integral and coefficients of Laurent series, and also the asymptotic representation of factorial, we obtain

$$\left|\frac{a_{-s}}{(s-1)!}\int_0^{\eta}(\eta-\zeta)^{s-1}\varphi(\zeta)\,d\zeta\right|<\sqrt{\frac{5}{4}}M\cdot\max|\varphi(\eta)|\cdot\frac{(\varrho e|\eta_{max}|)^{s}}{s^{s-\frac{1}{2}}}. \tag{1.277}$$

From (1.276) we have

$$\max|\varphi(\eta)|<\sqrt{\frac{4}{5}}\frac{e}{M}\frac{s^{s-\frac{1}{2}-\sigma}}{(\varrho e|\eta_{max}|)^{s}}=0\left[\left(\frac{s}{\varrho e|\eta_{max}|}\right)^{s-\frac{1}{2}-\sigma}\right]. \tag{1.278}$$

This be the unknown sufficient condition. All the functions integrated (in the sense of Riemann or Lebesgue) in domain Γ (page 39) and satisfying condition (1.278), form set ω_{L^-}.

Assuming now that $\varphi \in \omega_{L^-}$, we can interchange the position signs $\sum$ and $\int$ in (1.274), (1.275) and similar to them to formulas for $L^-(\xi, \beta)$. As a result we will obtain the following representation of the singular operators and operator-functions in the form of the series:

$$\boxed{\begin{aligned} L^-(\beta)\varphi &= \sum_{s=1}^{\infty} a_{-s}\beta^{-s}\varphi = \int_0^{\eta} \varphi(\zeta) \sum_{s=1}^{\infty} \frac{a_{-s}(\eta-\zeta)^{s-1}}{(s-1)!} d\zeta = \Phi(\eta), \\ L^-(\xi,\beta) &= \sum_{s=1}^{\infty} a_{-s}(\xi)\beta^{-s}\varphi = \int_0^{\eta} \varphi(\xi) \sum_{s=1}^{\infty} \frac{a_{-s}(\xi)(\eta-\zeta)^{s-1}}{(s-1)!} d\zeta = \Phi(\xi,\eta) \end{aligned}} \tag{1.279}$$

and similar expressions, which are obtained from (1.275).

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For the mixed operators in accordance with (1.263), (1.113), (1.143) and (1.279) let us have

$$\begin{aligned} L(\beta)\varphi = L^+(\beta)\varphi + L^-(\beta)\varphi = \sum_{s=-\infty}^{\infty} a_s\beta^s\varphi = \sum_{s=0}^{\infty} a_s\varphi^{(s)}(\eta) + \\ + \int_0^{\eta} \varphi(\zeta) \sum_{s=1}^{\infty} \frac{a_{-s}(\eta-\zeta)^{s-1}}{(s-1)!} d\zeta = \Phi(\eta) \end{aligned} \tag{1.280}$$

and the same expression, but with replacement of a by a (ξ) and $\Phi(\eta)$ on $\Phi(\xi,\eta)$ for L (ξ, β). Here $\varphi(\eta)$ must satisfy two conditions: (1.278) and (1.91).

2. Properties of the singular operators.

On the basis of (1.265), (1.269) and (1.279) and keeping in mind absolute and uniform convergence of series in (1.279), it is possible, in the same way as this was done into §5 and 6, to demonstrate [1] a series of the properties of the singular operators and operator-functions.

FOOTNOTE [1]. For the purpose of a decrease in the volume the majority of proofs let us lower. ENDFOOTNOTE.

a) If $\varphi(\eta)\in\omega_{L^-}$, that also $\Phi\in\omega_{L^-}$.

b) from p. a escape/ensues the existence of the product of the singular operators and operator-functions

$$L^-_{(1)}(\xi,\beta)\,L^-_{(2)}(\xi,\beta)\,\varphi = L^-_{(1)}(\xi,\beta)\,\{L^-_{(2)}(\xi,\beta)\,\varphi\}. \qquad (1.281)$$

c) from (1.269) follows the interchangeability of the product of the singular operators

$$L^-_{(1)}(\xi,\beta)\,L^-_{2}(\xi,\beta) = L^-_{(2)}(\xi,\beta)\,L^-_{1}(\xi,\beta). \qquad (1.282)$$

d) from (1.279) is evident that

$$L^-(\xi,\beta)\,\{0\} = 0. \qquad (1.283)$$

Therefore, in order that product (1.282) would become zero, it is sufficient if at least one operator is zero.

e) Singular operator is linear. It is more that,

$$L^-\left\{\sum_{n=1}^{\infty} A_n\varphi_n(\eta)\right\} = \sum_{n=1}^{\infty} A_n L^-\varphi_n(\eta), \qquad (1.284)$$

if in curly braces a series confronting converges evenly and absolutely, and $\varphi_n\in\omega_{L^-}$.

f) The singular operators, presented by series or closed expressions, can be store/added up, grouped and multiplied. This follows from the absolute convergence of series (1.257) and of absolute and uniform convergence of series (1.279). For illustration let us consider in detail a following example.

Function L $-$(z), generally speaking, can be represented in two forms: series (1.257) and the closed expression

$$L^-(z) = L_e^-(z) \qquad (1.285)$$

(e - the index of explicitness). [1].

FOOTNOTE [1]. Sometimes we can and not be able to find the closed expression, adequate to series (1.257), but then generally drops off the question concerning actions with the operators, presented in the closed form. ENDFOOTNOTE.

To this correspond operator's two representations:

$$L_e^-(\beta) \longleftrightarrow L_e^-(z) \equiv \sum_{s=1}^{\infty} a_{-s}\beta^{-s} \longleftrightarrow \sum_{s=1}^{\infty} a_{-s} z^{-s}. \qquad (1.286)$$

depicting one and the same operator, realized on (1.279).

Let, for example, four functions $\underbrace{L^-_{(1)}(z),\ L^-_{(2)}(z),\ L^-_{(3)}(z)}$ and $L^-_{(4)}(z)$, for which

$$L^-_{(j)}(z) = 0\left(\frac{1}{z^m}\right), \quad m > 1, \quad j = 1, 2, 3, 4, \qquad (1.287)$$

satisfy the relationship/ratio

$$L^-_{(1)}(z)\,L^-_{(2)}(z) + L^-_{(3)}(z) = L^-_{(4)}(z). \qquad (1.288)$$

This functional relationship/ratio is record/written in the

closed form:

$$L^-_{(1)\nu}(z)\,L^-_{(2)\nu}(z)+L^-_{(3)\nu}(z)=L^-_{(4)\nu}(z). \qquad (1.289)$$

Above property of the singular operators presented indicates that the escape/ensuing from (1.289) formal equality

$$L^-_{(1)\nu}(\beta)\,L^-_{(2)\nu}(\beta)+L^-_{(3)\nu}(\beta)=L^-_{(4)\nu}(\beta) \qquad (1.290)$$

expresses the real equality

$$(L^-_{(1)\nu}L^-_{(2)\nu}+L^-_{(3)\nu})\,\varphi(\eta)=L^-_{(4)}\varphi(\eta). \qquad (1.291)$$

Actually, since

$$L^-_{(j)\nu}(\beta)\,\varphi(\eta)=\int_0^\eta \varphi(\zeta)\sum_{s=1}^{\infty}\frac{a^{(j)}_{-s}(\eta-\zeta)^{s-1}}{(s-1)!}\,d\zeta, \qquad j=1,2,3,4, \qquad (1.292)$$

that

$$L^-_{(1)\nu}(\beta)\,L^-_{(2)\nu}(\beta)\,\varphi=L^-_{(1)\nu}\{L^-_{(2)\nu}\varphi\}=\int_0^\eta\left\{\int_0^\zeta \varphi(\zeta_1)\sum_{n=1}^{\infty}\frac{a^{(2)}_{-n}(\zeta-\zeta_1)^{n-1}}{(n-1)!}\times\right.$$
$$\left.\times\,d\zeta_1\right\}\sum_{s=1}^{\infty}\frac{a^{(1)}_{-s}(\eta-\zeta)^{s-1}}{(s-1)!}\,d\zeta. \qquad (1.293)$$

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Under integrals series confronting converge absolutely and evenly. In fact, after conducting the estimations, similar (1.277), let us have (see [24, page 69 and 44] (6.118)):

$$|a_{-s}| < MQ^s, \quad \frac{1}{(s-1)!} = \frac{s}{s!} < \sqrt{\frac{5}{4}} \cdot \frac{s^{1/2}}{e} \left(\frac{e}{s}\right)^s, \qquad (1.294a)$$

$$\left| \frac{a_{-s}^{(1)} (\eta - \zeta)^{s-1}}{(s-1)!} \right| < \sqrt{\frac{5}{4|\eta_{max}|}} MeQ \left(\frac{eQ|\eta_{max}|}{s} \right)^{s-\frac{1}{2}}.$$

Let us select $\underbrace{}_{s_0 > eQ|\eta_{max}|}$ and introduce the designations

$$A = \frac{M}{2|\eta_{max}|} \sqrt{5 s_0 eQ}, \quad q = \frac{eQ|\eta_{max}|}{s_0} < 1. \qquad (1.294b)$$

Then

$$\left| \frac{a_{-s}^{(1)} (\eta - \zeta)^{s-1}}{(s-1)!} \right| < Aq^s, \qquad (1.294c)$$

i.e. series are majorized by the infinitely decreasing progression. This proves the required convergence. Thus, entering in (1.293) series allow/assume multiplication, grouping and termwise integration. Therefore

$$L_{(1)\nu}^{-} L_{(2)\nu}^{-} \varphi = \int_0^\eta d\zeta \int_0^\zeta \varphi(\zeta_1) \sum_{s=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{-s}^{(1)} a_{-n}^{(2)} (\eta - \zeta)^{s-1} (\zeta - \zeta_1)^{n-1}}{(s-1)!\,(n-1)!} d\zeta_1. \qquad (1.295)$$

By varying the order of integration and after using formula (2.151) from [37], we will obtain

$$(L_{(1)\nu}^{-} L_{(2)\nu}^{-} + L_{(3)\nu}^{-}) \varphi = \int_0^\eta \varphi(\zeta) \left\{ \sum_{p=2}^{\infty} \left[\left(\sum_{s=1}^{p-1} a_{-s}^{(1)} a_{-(p-s)}^{(2)} + a_{-p}^{(3)} \right] \times \right. \right.$$

$$\left. \times \frac{(\eta - \zeta)^{p-1}}{(p-1)!} + a_{-1}^{(3)} \right\} d\zeta. \qquad (1.296)$$

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Now, keeping in mind absolute convergence of series (1.257), let us find

$$L^{-}_{(4)}(z) = L^{-}_{(1)}(z)\, L^{-}_{(2)}(z) + L^{-}_{(3)}(z) = \sum_{p=1}^{\infty} a^{(1)}_{-p} z^{-p} \cdot \sum_{n=1}^{\infty} a^{(2)}_{-n} z^{-n} + \sum_{p=1}^{\infty} a^{(3)}_{-p} z^{-p} =$$
$$= \sum_{p=2}^{\infty} \left[\left(\sum_{s=1}^{p-1} a^{(1)}_{-s} a^{(2)}_{-(p-s)} \right) + a^{(3)}_{-p} \right] z^{-p} + a^{(3)}_{-1} z^{-1}, \qquad (1.297)$$

whence

$$a^{(4)}_{-p} = \left(\sum_{s=1}^{p-1} a^{(1)}_{-s} a^{(2)}_{-(p-s)} \right) + a^{(3)}_{-p} \ (p > 2), \quad a^{(4)}_{-1} = a^{(3)}_{-1}. \qquad (1.298)$$

Finally, comparing (1.296) with (1.292) and taking into account (1.298), we see that really/actually it occurs (1.291).

For simplicity of notation we examined singular operators. It is

not difficult to ascertain that all lining/calculations and conclusions will remain in force and for the operator-function, when $a_{-s} = a_{-s}(\xi)$. For example,

$$\mathrm{ch}\, i\left(\frac{\pi}{2}+\frac{\xi}{\beta}\right)\{\varphi\} = -\sin\frac{\xi}{\beta}\{\varphi\}. \qquad (1.299)$$

However, examining operator-function, it is necessary to follow the interval of the variation ξ. So, the equality

$$\cos^2\frac{1}{\xi\beta}\{\varphi\} - \sin^2\frac{1}{\xi\beta}\{\varphi\} = \cos\frac{2}{\xi\beta}\{\varphi\} \qquad (1.300)$$

will be valid only in interval/gap $[\xi_0, 1]$, $\xi_0 > 0$, since in (1.139) series they will converge unevenly.

g) Any rational relatively β proper fraction represents the singular operator

$$L^-(\beta) = \frac{A_p\beta^p + A_{p-1}\beta^{p-1} + \ldots + A_1\beta + A_0}{B_n\beta^n + B_{n-1}\beta^{n-1} + \ldots + B_1\beta + B_0}, \quad p < n. \qquad (1.301)$$

The sum and the product of proper fractions give again proper fraction. Therefore on the basis p. g it is possible to claim that above operational fractions it is possible to produce the actions of addition, multiplication, resolution into partial fractions, etc. For example,

$$\left.\begin{aligned}&\frac{\beta}{1-\beta^2}\varphi+\frac{1}{1+\beta}\varphi-\frac{2}{\beta}\varphi=\frac{2\beta^2+\beta-2}{\beta(1-\beta^2)}\varphi,\\&\frac{1}{1-\beta^2}\left\{\frac{1-\beta}{(1+\beta)^2}\varphi\right\}=\frac{1}{(1+\beta)^3}\varphi,\\&\frac{1}{\xi\beta}\{\varphi\}+\frac{2\xi\beta}{1+\beta^2}\{\varphi\}=\frac{1+(1+2\xi^2)\beta^2}{\xi\beta(1+\beta^2)}\{\varphi\}.\end{aligned}\right\}\quad(1.302)$$

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In a last/latter example $\xi\in[\xi_0, 1]$, $\xi_0>0$, since in interval/gap (1.139) a series converges unevenly.

Let us note still that on the strength of the permutability of the regular and singular operators thus far let us consider inadmissible this representation of fractions as, for example,

$$\frac{1+\beta}{1+\beta^2}=(1+\beta)\cdot\frac{1}{1+\beta^2}.\quad(1.303)$$

h) If function $L^-(z)\longleftrightarrow L^-(\beta)$ is multisheeted, then let us agree to always choose those expansions (1.257) and (1.258), which correspond to the so-called main branch of function, for example,

$$\ln\frac{\beta+1}{\beta-1}=2\sum_{s=1}^{\infty}\frac{1}{(2s-1)\beta^{2s-1}},$$

$$\frac{\pi}{2}-\operatorname{arctg}\beta=\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)\beta^{2s+1}},$$

$$\frac{1}{\sqrt{\beta^2-k^2}}=\frac{1}{\beta}\left(1+\frac{1}{2}\frac{k^2}{\beta^2}+\frac{1\cdot 3}{2\cdot 4}\frac{k^4}{\beta^4}+\dots\right)=\frac{1}{\beta}+\sum_{s=1}^{\infty}\frac{(2s-1)!!}{(2s)!!}\cdot\frac{k^{2s}}{\beta^{2s+1}}.\quad(1.304)$$

Then and to these operators will be spread the enumerated previously properties, in particular p. f and g. So that, for example,

$$\frac{1}{\xi\sqrt{\beta^2-\xi^2}}\varphi-\frac{\sqrt{\beta^2-\xi^2}}{\beta^2+\xi^2}\varphi=$$
$$=\frac{(1-\xi)\beta^2+(1+\xi)\xi}{\xi(\beta^2+\xi^2)\sqrt{\beta^2-\xi^2}}\varphi,\ \xi\in[\xi_0,1],\ \xi_0>0. \qquad (1.305)$$

i) Taking into account the infinite differentiability of series (1.257), in the same way as this was done in p. 3 of §5, we come to the conclusion that singular operator unlimitedly "we differentiate" with respect to symbol β.

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According to (1.279),

$$\frac{\partial L^-}{\partial\beta}\varphi=-\sum_{s=2}^{\infty}(s-1)a_{-(s-1)}\beta^{-s}\varphi=-\int_0^{\eta}\varphi(\zeta)\sum_{s=2}^{\infty}\frac{a_{-(s-1)}(\eta-\zeta)^{s-1}}{(s-2)!}d\zeta.$$

$$\frac{\partial^m L^-}{\partial\beta^m}\varphi=(-1)^m\sum_{s=m+1}^{\infty}(s-1)(s-2)\ldots(s-m)a_{-(s-m)}\beta^{-s}\varphi=$$

$$=(-1)^m\int_0^{\eta}\varphi(\zeta)\sum_{s=m+1}^{\infty}\frac{a_{-(s-m)}(\eta-\zeta)^{s-1}}{(s-m-1)!}\,d\zeta \quad (1.306)$$

it is possible to show, as this was done above, that if the function ϕ was limited and integrated, then series here also converge absolutely and evenly.

Now, besides addition and multiplication, into the number of permissible actions above the singular operators it is possible to include/connect differentiation. For example,

$$\left(\frac{\partial}{\partial\beta}\frac{\beta}{\sqrt{\xi^4\beta^4+1}}\right)(\varphi)=\frac{1-\xi^4\beta^4}{\sqrt{(\xi^4\beta^4+1)^3}}\varphi,\ \xi\in[\xi_0,1],\ \xi_0>0. \quad (1.307)$$

More that, it is possible to speak about differential equations for the singular operators. So, the operator

$$L^-(\xi,\beta)=e^{\sqrt{\xi^2\beta^2-k^2}} \quad (1.308)$$

is the solution of the problem:

$$(\xi^2\beta^2-k^2)\left(\frac{\partial L^-}{\partial\beta}\right)^2-\xi^4\beta^2(L^-)^2=0,\ L^-\left(\xi,\frac{k}{\xi}\right)=1. \quad (1.309)$$

Is appropriate to call to mind the observation, done at the end p. 3 § 5 apropos of terms "differentiation with respect to β", "expansion in a series according to degrees β" and so forth, it here retains its force.

j) Let us designate by Γ_ξ that range of change in the real or complex variable ξ, in which function L (ξ, z) is regular in vicinity z = ∞ (i.e. correctly expansion (1.257)). Then functions $a_{-s}(\xi)$ are limited with any fixed/recorded s (although they can unlimitedly grow/rise with s $\to \infty$), since (see (1.293))

$$|a_{-s}(\xi)| < M\varrho^s, \ \xi \in \Gamma_\xi \tag{1.310}$$

(relative to ϱ amd M see page 71). If moreover:

1) functions $a_{-s}(\xi)$ are evenly continuous in Γ_ξ, then operator-function L^- (ξ, β) is continuous:

$$\boxed{\lim_{\xi\to\xi_1} \Phi(\xi, \eta) \lim_{\xi\to\xi_1} L^-(\xi, \beta)\varphi = L^-(\xi_1, \beta) = \Phi(\xi_1, \eta).} \tag{1.311}$$

or, otherwise,

$$\lim_{\xi\to\xi_1} \int_0^\eta \varphi(\zeta) \sum_{s=1}^{\infty} \frac{a_{-s}(\xi)(\eta-\zeta)^{s-1}}{(s-1)!} d\zeta = \int_0^\eta \varphi(\zeta) \sum \frac{a_{-s}(\xi_1)(\eta-\zeta)^{s-1}}{(s-1)!} d\zeta; \tag{1.312}$$

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2) functions $a_{-s}(\xi)$ are integrated and their integrals are evenly continuous in Γ_ξ, that L^- (ξ, β) is integrated, whereupon

$$\boxed{\int \Phi(\xi, \eta)\, d\xi = \int L^-(\xi, \beta)\varphi(\eta)\, d\xi = \left(\int L^-(\xi, \beta)\, d\xi\right)\{\varphi(\eta)\}.} \tag{1.313}$$

cr, otherwise,

$$\int\left\{\int_0^\eta \varphi(\zeta)\sum_{s=1}^{\infty}\frac{a_{-s}(\xi)(\eta-\zeta)^{s-1}}{(s-1)!}\,d\zeta\right\}d\xi =$$

$$=\int_0^\eta \varphi(\zeta)\left(\sum_{s=1}^{\infty}\frac{(\eta-\zeta)^{s-1}}{(s-1)!}\int a_{-s}(\xi)\,d\xi\right)d\zeta; \qquad (1.314)$$

3) functions $a_{-s}(\xi)m$ once are differentiated and all their derivatives are evenly continuous in Γ_ξ, then operator-function $L^-(\xi, \beta)$ is differentiated on ξ, whereupon

$$\boxed{\frac{\partial^m\Phi(\xi,\eta)}{\partial\xi^m}=\frac{\partial^m}{\partial\xi^m}[L^-(\xi,\beta)\varphi(\eta)]=\frac{d^mL^-}{d\xi^m}\varphi=L^{(m)}\varphi,} \qquad (1.315)$$

cr, otherwise,

$$\frac{\partial^m}{\partial\xi^m}\int_0^\eta \varphi(\zeta)\sum_{s=1}^{\infty}\frac{a_{-s}(\xi)(\eta-\zeta)^{s-1}}{(s-1)!}\,d\zeta =$$

$$=\int_0^\eta \varphi(\zeta)\sum_{s=1}^{\infty}\frac{a_{-s}^{(m)}(\xi)(\eta-\zeta)^{s-1}}{(s-1)!}\,d\zeta. \qquad (1.316)$$

The proofs of the enumerated properties (they are based on absolute convergence of series (1.257) and on absolute and uniform

convergence of series (1.279)) lower, since they are analogous thereby, that are carried out in p. 4 §5; let us give only several examples.

1. If

$$\frac{\sqrt{\beta^2+\xi\beta+1}}{\beta^2-\xi^2}\varphi=\Phi(\xi,\eta). \quad (1.317)$$

that

$$\frac{\sqrt{\beta^2+1}}{\beta^2}\varphi=\Phi(0,\eta); \quad (1.318)$$

but, for example, $\lim\limits_{\xi\to 0}\frac{1}{1-\xi\beta}\varphi(\eta)$ does not exist, since, function $a_{-1}(\xi)=-\frac{1}{\xi^2}$ is disruptive with $\xi = 0$.

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2. If

$$\frac{1}{\xi-\beta}\varphi=\Phi(\xi,\eta). \quad (1.319)$$

that

$$\frac{\xi}{\beta(\beta-\xi)}\varphi(\eta)=\int\limits_0^\xi\Phi(\zeta,\eta)d\zeta,\ \frac{1}{\beta(\beta-1)}\varphi(\eta)=\int\limits_0^1\Phi(\xi,\eta)d\xi. \quad (1.320)$$

3. The general solution of the equation

$$\beta^2L''-L=0 \quad (1.321)$$

will be

$$L^{-}(\xi,\beta)=A\operatorname{ch}\frac{\xi}{\beta}+B\operatorname{sh}\frac{\xi}{\beta}. \quad (1.322)$$

where A = A (β) and B = B (β) - some operators.

§8. Realization of the singular operators.

The present paragraph almost wholly includes lead-in part §6; therefore to here inexpediently repeat there the reasonings presented.

1. Method for analytic functions.

Let us consider that $\eta=0\in\Gamma$ (see Section e page 39), and let us narrow down the class of the permissible functions $\varphi(\eta)$ to ω (see Section 7 pages 44): $\varphi\in\omega$ with $\eta\in\Gamma$. Then ϕ will be analytic function, and that means correctly the expansion

$$\varphi(\eta)=c_0+c_1\eta^2+c_2\eta^2+\ldots=\sum_{n=0}^{\infty}c_n\eta^n. \qquad (1.323)$$

Since $\omega\subset\omega_L$ (see page 71), to such functions ϕ it is possible to apply the singular operators

$$L(\xi,\beta)=L^-(\xi,\beta)=\frac{a_{-1}(\xi)}{\beta}+\frac{a_{-2}(\xi)}{\beta^2}+\ldots=\sum_{s=1}^{\infty}a_{-s}(\xi)\beta^{-s}. \qquad (1.324)$$

as a result of which we will obtain

$$L(\xi,\beta)\varphi(\eta)-L^{-}(\xi,\beta)\varphi(\eta)=\Phi(\xi,\eta). \qquad (1.325)$$

as is known, the integration (including multiple) of functions $\in\omega$ dces not derive/conclude beyond limits ω. Therefore

$$\Phi(\xi,\eta)=A_0(\xi)+A_1(\xi)\eta+\ldots=\sum_{p=0}^{\infty}A_p(\xi)\eta^p. \qquad (1.326)$$

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From (1.265)

$$\beta^{-s}\eta^{n}=\frac{1}{(s-1)!}\int_0^{\eta}(\eta-\zeta)^{s-1}\zeta^{n}d\zeta=\frac{n!}{(s+n)!}\eta^{s+n}. \qquad (1.327)$$

By introducing (1.323), (1.324) and (1.326) in (1.325) and by taking into account (1.327), let us find

$$\left.\begin{aligned}
&A_0=0,\\
&A_1=a_{-1}c_0,\\
&A_2=\frac{1}{2!}(a_{-1}c_1+a_{-2}c_0),\\
&A_3=\frac{1}{3!}(2!a_{-1}c_2+1!a_{-2}c_1+0!a_{-3}c_0),\\
&A_4=\frac{1}{4!}(3!a_{-1}c_3+2!a_{-2}c_2+1!a_{-3}c_1+0!a_{-4}c_0),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \qquad (1.328)$$

i. e.,

$$A_0 = 0,\ A_p(\xi) = \frac{1}{p!}\sum_{s=1}^{p}(p-s)!\,a_{-s}(\xi)\,c_{p-s},\ p = 1, 2, \ldots \quad (1.329)$$

Example. Let us find

$$\frac{1}{2}\ln\frac{\beta+1}{\beta-1}\{\operatorname{sh}\eta\} = \Phi(\eta). \quad (1.330)$$

In this case (see (1.304))

$$a_{-2s} = 0,\ a_{-(2s-1)} = \frac{1}{2s-1};\ c_{2n} = 0,\ c_{2n-1} = \frac{1}{(2n-1)!},$$
$$A_{2p-1} = 0,\ A_{2p} = \left(1 + \frac{1}{3} + \ldots + \frac{1}{2p-1}\right)\frac{1}{(2p)!}$$

and consequently,

$$\Phi(\eta) = \frac{\eta^2}{2!} + \frac{4}{3}\frac{\eta^4}{4!} + \frac{23}{15}\frac{\eta^6}{6!} + \frac{176}{105}\frac{\eta^8}{8!} + \ldots =$$
$$= \sum_{p=1}^{\infty}\left(1 + \frac{1}{3} + \ldots + \frac{1}{2p-1}\right)\frac{\eta^{2p}}{(2p)!}. \quad (1.331)$$

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2. Realization above polynomials, exponential, hyperbolic and

trigonometric functions.

a) Let

$$\varphi(\eta)=c_0+c_1\eta+c_2\eta^2+\ldots+c_N\eta^N. \quad (1.332)$$

From (1.326) and (1.326) we obtain as for (1.324),

$$L^-(\beta)\varphi(\eta)=\sum_{p=1}^{\infty}\left(\sum_{n=0}^{p-1:N} n!a_{-(p-n)}c_n\right)\frac{\eta^p}{p!}, \quad (1.333)$$

where n it varies from 0 to p-1, if p-1 < N, and from 0 to N, if p-1 ≥ N.

b) Let

$$\varphi(\eta)=e^{m\eta}. \quad (1.334)$$

Since

$$\int_0^{\eta}(\eta-\zeta)^{s-1}e^{m\zeta}d\zeta=\frac{(s-1)!}{m^s}\left[e^{m\eta}-\sum_{n=0}^{s-1}\frac{(m\eta)^s}{s!}\right] \quad (1.335)$$

(this formula is obtained by repeated integration in parts), of (1.279) let us find

$$\boxed{L^-(\xi,\beta)e^{m\eta}=e^{m\eta}L^-(\xi,m)-\sum_{p=0}^{\infty}\left(\sum_{s=p+1}^{\infty}\frac{a_{-s}(\xi)}{m^s}\right)\frac{(m\eta)^p}{p!}}. \quad (1.336)$$

(m - any).

Comparing this formula with (1.192), is detected one of the most important differences of the singular operators of the regular.

Without selecting first term in (1.336), it is possible to write

$$\boxed{L^{-}(\xi,\beta)\,e^{m\eta}=\sum_{p=1}^{\infty}\left(\sum_{s=1}^{p}\frac{a_{-s}(\xi)}{m^{s}}\right)\frac{(m\eta)^{p}}{p!}.} \qquad (1.337)$$

c) substituting in (1.336) m on - m and taking the linear combination of results, we obtain

$$L^{-}(\xi,\beta)\,{\mathrm{ch} \atop \mathrm{sh}}\,m\eta=\frac{1}{2}[L^{-}(\xi,m)+L^{-}(\xi,-m)]\,{\mathrm{ch} \atop \mathrm{sh}}\,m\eta+$$
$$+\frac{1}{2}[L^{-}(\xi,m)-L^{-}(\xi,-m)]\,{\mathrm{sh} \atop \mathrm{ch}}\,m\eta-\Delta_{{c \atop s}}(\eta,\xi,m), \qquad (1.338)$$

where markedly

$$\Delta_{c}(\eta,\xi,m)=\sum_{p=0}^{\infty}\left(\sum_{n=p+1}^{\infty}\frac{1+(-1)^{n+p}}{2}\frac{a_{-n}(\xi)}{m^{n}}\right)\frac{(m\eta)^{p}}{p!},$$
$$\Delta_{s}(\eta,\xi,m)=\sum_{p=0}^{\infty}\left(\sum_{n=p+1}^{\infty}\frac{1-(-1)^{n+p}}{2}\frac{a_{-n}(\xi)}{m^{n}}\right)\frac{(m\eta)^{p}}{p!}. \qquad (1.339)$$

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Hence, in particular:

if operator $L^{-}(\xi,\beta)$ even relatively β, then

$$\boxed{\begin{aligned}L^{-}(\xi,\beta)\,\mathrm{ch}\,m\eta&=L^{-}(\xi,m)\,\mathrm{ch}\,m\eta-\sum_{p=0}^{\infty}\left(\sum_{n=p+1}^{\infty}\frac{a_{-2n}}{m^{2n}}\right)\frac{(m\eta)^{2p}}{(2p)!},\\ L^{-}(\xi,\beta)\,\mathrm{sh}\,m\eta&=L^{-}(\xi,m)\,\mathrm{sh}\,m\eta-\sum_{p=0}^{\infty}\left(\sum_{n=p+1}^{\infty}\frac{a_{-2n}}{m^{2n}}\right)\frac{(m\eta)^{2p+1}}{(2p+1)!};\end{aligned}} \qquad (1.340)$$

if operator $L^{-}(\xi, \beta)$ odd relatively β, then

$$\boxed{\begin{aligned} &L^{-}(\xi, \beta)\,\mathrm{ch}\,m\eta = L^{-}(\xi, m)\,\mathrm{sh}\,m\eta - \sum_{p=0}^{\infty}\left(\sum_{n=p+1}^{\infty}\frac{a_{-(2n+1)}}{m^{2n+1}}\right)\frac{(m\eta)^{2p+1}}{(2p+1)!},\\ &L^{-}(\xi, \beta)\,\mathrm{sh}\,m\eta = L^{-}(\xi, m)\,\mathrm{ch}\,m\eta - \sum_{p=0}^{\infty}\left(\sum_{n=p}^{\infty}\frac{a_{-(2n+1)}}{m^{2n+1}}\right)\frac{(m\eta)^{2p}}{(2p)!}.\end{aligned}} \quad (1.341)$$

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By substituting m on im, let us have

$$\begin{aligned} L^{-}(\xi, \beta)\cos m\eta &= \frac{1}{2}[L^{-}(\xi, im) + L^{-}(\xi, -im)]\cos m\eta + \\ &+ \frac{i}{2}[L^{-}(\xi, im) - L^{-}(\xi, -im)]\sin m\eta - \Delta_c(\eta, \xi, im), \\ L^{-}(\xi, \beta)\sin m\eta &= \frac{1}{2}[L^{-}(\xi, im) + L^{-}(\xi, -im)]\sin m\eta - \\ &- \frac{i}{2}[L^{-}(\xi, im) - L^{-}(\xi, -im)]\cos m\eta + i\Delta_s(\eta, \xi, im)\end{aligned} \quad (1.342)$$

and further, if operator $L^{-}(\xi, \beta)$ even relatively β, then

$$\boxed{\begin{aligned} &L^{-}(\xi, \beta)\cos m\eta = L^{-}(\xi, im)\cos m\eta - \\ &-\sum_{p=0}^{\infty}\left(\sum_{n=p+1}^{\infty}(-1)^n\frac{a_{-2n}}{m^{2n}}\right)(-1)^p\frac{(m\eta)^{2p}}{(2p)!},\\ &L^{-}(\xi, \beta)\sin m\eta = L^{-}(\xi, im)\sin m\eta - \\ &-\sum_{p=0}^{\infty}\left(\sum_{n=p+1}^{\infty}(-1)^n\frac{a_{-2n}}{m^{2n}}\right)(-1)^p\frac{(m\eta)^{2p+1}}{(2p+1)!}.\end{aligned}} \quad (1.343)$$

if operator $L^-(\xi, \beta)$ odd relatively β, then

$$\begin{gathered} L^-(\xi,\beta)\cos m\eta = iL^-(\xi, im)\sin m\eta - \\ -\sum_{p=0}^{\infty}\Bigl(\sum_{n=p+1}^{\infty}(-1)^n\frac{a_{-(2n+1)}}{m^{2n+1}}\Bigr)(-1)^p\frac{(m\eta)^{2p+1}}{(2p+1)!}, \\ L^-(\xi,\beta)\sin m\eta = -iL^-(\xi, im)\cos m\eta + \\ +\sum_{p=0}^{\infty}\Bigl(\sum_{n=p}^{\infty}(-1)^n\frac{a_{-(2n+1)}}{m^{2n+1}}\Bigr)(-1)^p\frac{(m\eta)^{2p}}{(2p)!}. \end{gathered} \tag{1.344}$$

If we in $L^-(\xi, \beta)$ $\operatorname{ch} m\eta$ and so forth do not select terms, similar $L^-(\xi, m)\operatorname{ch} m\eta$ and so forth (see (1.203) - (1.209)), then, after using (1.337), it is possible to obtain the following expressions:

$$\begin{gathered} L^-(\xi,\beta)\operatorname{ch} m\eta = \sum_{p=1}^{\infty}\Bigl[\sum_{s=1}^{p}\frac{1+(-1)^{s+p}}{2}\frac{a_{-s}(\xi)}{m^s}\Bigr]\frac{(m\eta)^p}{p!} = \\ = \sum_{p=1}^{\infty}\Bigl(\sum_{s=1}^{p}\frac{a_{-2s}(\xi)}{m^{2s}}\Bigr)\frac{(m\eta)^{2p}}{(2p)!} + \sum_{p=0}^{\infty}\Bigl(\sum_{s=0}^{p}\frac{a_{-(2s+1)}(\xi)}{m^{2s+1}}\Bigr)\frac{(m\eta)^{2p+1}}{(2p+1)!}, \\ L^-(\xi,\beta)\operatorname{sh} m\eta = \sum_{p=1}^{\infty}\Bigl[\sum_{s=1}^{p}\frac{1-(-1)^{s+p}}{2}\frac{a_{-s}(\xi)}{m^s}\Bigr]\frac{(m\eta)^p}{p!} = \\ = \sum_{p=1}^{\infty}\Bigl(\sum_{s=1}^{p}\frac{a_{-2s}(\xi)}{m^{2s}}\Bigr)\frac{(m\eta)^{2p+1}}{(2p+1)!} + \sum_{p=1}^{\infty}\Bigl(\sum_{s=0}^{p-1}\frac{a_{-(2s+1)}(\xi)}{m^{2s+1}}\Bigr)\frac{(m\eta)^{2p}}{(2p)!}, \end{gathered} \tag{1.345}$$

whence

$$\begin{gathered} L^-(\xi,\beta)\cos m\eta = \sum_{p=1}^{\infty}\Bigl(\sum_{s=1}^{p}(-1)^s\frac{a_{-2s}(\xi)}{m^{2s}}\Bigr)(-1)^p\cdot\frac{(m\eta)^{2p}}{(2p)!} + \\ + \sum_{p=0}^{\infty}\Bigl(\sum_{s=0}^{p}(-1)^s\frac{a_{-(2s+1)}(\xi)}{m^{2s+1}}\Bigr)(-1)^p\frac{(m\eta)^{2p+1}}{(2p+1)!}, \\ L^-(\xi,\beta)\sin m\eta = \sum_{p=1}^{\infty}\Bigl(\sum_{s=1}^{p}(-1)^s\frac{a_{-2s}(\xi)}{m^{2s}}\Bigr)(-1)^p\times \\ \times\frac{(m\eta)^{2p+1}}{(2p+1)!} + \sum_{p=1}^{\infty}\Bigl(\sum_{s=0}^{p}(-1)^{s+1}\frac{a_{-(2s+1)}(\xi)}{m^{2s+1}}\Bigr)(-1)^p\frac{(m\eta)^{2p}}{(2p)!}. \end{gathered} \tag{1.346}$$

Let us note that these formulas, as (1.337), have considerable advantage over (1.336) and (1.339) - (1.344), since of the latter the internal sums are infinite and in certain cases (for example, p. 6b the present paragraph) they can turn out to be those which are diverging.

Operator $(\beta+k)^{-n}$

Let us consider the operator

$$L^{-}(\beta)=\frac{1}{(\beta-k)^{n}}=(\beta-k)^{-n},\ n>1\text{—integer.} \quad (1.347)$$

This operator singular, since the function $L(z)=(z-k)^{n}$ has a pole at point k, but is regular at infinity. Therefore occurs the expansion

$$L^{-}(\beta)=(\beta-k)^{-n}=\frac{1}{\beta^{n}}\left[1+\sum_{p=1}^{\infty}\frac{n(n+1)\ldots(n+p-1)}{p!}\frac{k^{p}}{\beta^{p}}\right]=$$
$$=\frac{1}{\beta^{n}}+\sum\frac{(s-1)!}{(n-1)!\,(s-n)!}\cdot\frac{k^{s-n}}{\beta^{s}}. \quad (1.348)$$

whence

$$a_{-s}=0,\ \text{если}\ s<n;\ a_{-n}=1,$$
$$a_{-s}=\frac{(s-1)!}{(n-1)!\,(s-n)!}k^{s-n},\ \text{если}\ s>n. \quad (1.349)$$

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Taking into account this, we convert the series, entering in (1.279):

$$\sum_{s=1}^{\infty}\frac{a_{-s}(\eta-\zeta)^{s-1}}{(s-1)!}=\frac{(\eta-\zeta)^{n-1}}{(n-1)!}+\frac{1}{(n-1)!}\sum_{s=n+1}^{\infty}\frac{(\eta-\zeta)^{s-1}k^{s-n}}{(s-n)!}=$$
$$=\frac{(\eta-\zeta)^{n-1}}{(n-1)!}\left[1+\sum_{p=1}^{\infty}\frac{(\eta-\zeta)^{p}k^{p}}{p!}\right]=\frac{(\eta-\zeta)^{n-1}}{(n-1)!}e^{k(\eta-\zeta)}. \quad (1.350)$$

By introducing (1.350) in (1.279), we will obtain

$$\boxed{\frac{1}{(\beta-k)^{n}}\varphi(\eta)=(\beta-k)^{-n}\varphi(\eta)=\frac{e^{k\eta}}{(n-1)!}\int_{0}^{\eta}e^{-k\zeta}(\eta-\zeta)^{n-1}\varphi(\zeta)\,d\zeta,}$$

$n \geqslant 1$, whole, (1.351)

or, after replacing k by-k,

$$\frac{1}{(\beta+k)^{n}}\varphi(\eta)=(\beta+k)^{-n}\varphi(\eta)=\frac{e^{-k\eta}}{(n-1)!}\int_{0}^{\eta}e^{k\zeta}(\eta-\zeta)^{n-1}\varphi(\zeta)\,d\zeta. \quad (1.352)$$

$n \geqslant 1$, whole.

Specifically, with $n = 1$

$$\boxed{\frac{1}{\beta-k}\varphi(\eta)=(\beta-k)^{-1}\varphi(\eta)=e^{k\eta}\int_{0}^{\eta}e^{-k\zeta}\varphi(\zeta)\,d\zeta.} \quad (1.353)$$

4. Operator (L^- (ξ, β + k).

According to determination (1.259)

$$L^-(\xi, \beta + k) = \sum_{n=1}^{\infty} a_{-n}(\xi)(\beta + k)^{-n}. \qquad (1.354)$$

Since the operators $(\beta + k)^{-n}$ are singular (see Section 3), also L^- (ξ, β) singular. Keeping in mind (1.351), we obtain

$$L^-(\xi, \beta + k)\varphi(\eta) = e^{-k\eta}\int_0^{\eta} e^{k\zeta}\varphi(\zeta)\sum_{n=1}^{\infty}\frac{a_{-n}(\xi)(\eta-\zeta)^{n-1}}{(n-1)!}\,d\zeta. \qquad (1.355)$$

The legitimacy of the exchange of addition and integration follows from p. 2a §7.

Comparing now (1.355) with (1.279), is detected, that

$$\boxed{L^-(\xi, \beta + k)\varphi(\eta) = e^{-k\eta}L^-(\xi, \beta)\,[e^{k\eta}\varphi(\eta)].} \qquad (1.356)$$

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Thus, formula (1.225) turns out to be accurate both for regular and for the singular operators.

5. Operator $\frac{\partial L^-(\xi, \beta)}{\partial \beta}$

According to (1.257), (1.259) and (1.279) we have (we produce exchange in roll)

$$\frac{\partial L^{-}(\xi,\beta)}{\partial\beta}\varphi(\eta)=-\sum_{s=1}^{\infty}\frac{sa_{-s}(\xi)}{\beta^{s+1}}\varphi(\eta)=-\int_{0}^{\eta}\varphi(\eta-\zeta)\sum_{s=1}^{\infty}\frac{s}{s!}a_{-s}(\xi)\zeta^{s}d\zeta=$$
$$=-\int_{0}^{\eta}\varphi(\eta-\zeta)\sum_{s=1}^{\infty}\frac{a_{-s}(\xi)\zeta^{s}}{(s-1)!}d\zeta. \quad (1.357)$$

On the other hand, on (1.279)

$$L^{-}(\xi,\beta)\{\eta\varphi(\eta)\}=\int_{0}^{\eta}(\eta-\zeta)\varphi(\eta-\zeta)\sum_{s=1}^{\infty}\frac{a_{-s}(\xi)\zeta^{s-1}}{(s-1)!}d\zeta=$$
$$=\eta\int_{0}^{\eta}\varphi(\eta-\zeta)\sum_{s=1}^{\infty}\frac{a_{-s}(\xi)\zeta^{s-1}}{(s-1)!}d\zeta-\int_{0}^{\eta}\varphi(\eta-\zeta)\sum_{s=1}^{\infty}\frac{a_{-s}(\xi)\zeta^{s}}{(s-1)!}d\zeta. \quad (1.358)$$

Comparing both formulas, we see that

$$\boxed{\frac{\partial L^{-}(\xi,\beta)}{\partial\beta}\{\varphi(\eta)\}=L^{-}(\xi,\beta)\{\eta\varphi(\eta)\}-\eta L^{-}(\xi,\beta)\{\varphi(\eta)\}.} \quad (1.359)$$

This formula coincides with (1.232). Since series $\frac{d^{n}L^{-}(\xi,z)}{dz^{n}}$ converge with any n, repeatedst possible application/use (1.359), as a result of which we will obtain (1.233) and generally

$$\boxed{\frac{\partial^{n}L(\xi,\beta)}{\partial\beta^{n}}\varphi(\eta)=\sum_{s=0}^{n}(-1)^{s}C_{n}^{s}\eta^{s}L(\xi,\beta\{\eta^{n-s}\varphi(\eta)\}.} \quad (1.360)$$

whereupon this formula is accurate both for singular and for the regular operators.

6. Examples of realization [1].

FOOTNOTE [1]. The large part of the given here results was obtained together with N. A. Wenzel. ENDFOOTNOTE.

Here in essence will be given examples of the determination of the values of the operators above concrete/specific/actual functions and, furthermore, the realization of some new operators.

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a) On the basis of the known integrals

$$\left.\begin{aligned}
&\int_0^x e^{at}dt = \frac{1}{a}(e^{ax}-1),\\
&\int_0^x te^{at}dt = \frac{1}{a^2}[e^{ax}(ax-1)+1],\\
&\int_0^x t^2e^{at}dt = \frac{2!}{a^3}\left[e^{ax}\left(\frac{a^2x^2}{2!}-ax+1\right)-1\right],\\
&\int_0^x t^ne^{at}dt = (-1)^n\frac{n!}{a^{n+1}}\left[e^{ax}\sum_{p=0}^{n}(-1)^p\frac{(ax)^p}{p!}-1\right]
\end{aligned}\right\} \quad (1.361)$$

and taking into account (1.353), let us find

$$\boxed{\begin{aligned}&\frac{1}{\beta-k}\{1\}=\frac{1}{k}(e^{k\eta}-1),\quad \frac{1}{\beta-k}\{\eta\}=\frac{1}{k^2}(e^{k\eta}-1-k\eta),\\ &\frac{1}{\beta-k}\{\eta^2\}=\frac{2!}{k^3}\left(e^{k\eta}-1-k\eta-\frac{k^2\eta^2}{2!}\right),\\ &\frac{1}{\beta-k}\{\eta^n\}=\frac{n!}{k^{n+1}}\left(e^{k\eta}-1-k\eta-\frac{k^2\eta^2}{2!}-\ldots-\frac{k^n\eta^n}{n!}\right).\end{aligned}} \quad (1.362)$$

b) Using formula (1.353), we obtain

$$\boxed{\begin{aligned}&\frac{1}{\beta-k}e^{m\eta}=\frac{e^{m\eta}-e^{k\eta}}{m-k},\\ &\frac{1}{\beta-k}\operatorname{ch}m\eta=\frac{k\operatorname{ch}m\eta+m\operatorname{sh}m\eta-ke^{k\eta}}{m^2-k^2},\\ &\frac{1}{\beta-k}\operatorname{sh}m\eta=\frac{k\operatorname{sh}m\eta+m\operatorname{ch}m\eta-me^{k\eta}}{m^2-k^2},\\ &\frac{1}{\beta-k}\cos m\eta=\frac{-k\cos m\eta+m\sin m\eta+ke^{k\eta}}{m^2+k^2},\\ &\frac{1}{\beta-k}\sin m\eta=\frac{-k\sin m\eta-m\cos m\eta+me^{k\eta}}{m^2+k^2}.\\ &\qquad\qquad\qquad m\neq k.\end{aligned}} \quad (1.363)$$

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To this same result it was possible to arrive on the basis p. 2b. In this case it was necessary to use formulas (1.337), (1.345) and (1.346). But if we apply (1.336), (1.338), (1.339) and (1.342), then it is necessary to introduce supplementary, that not being internally necessary for operations (1.363) condition $|k| < |m|$; otherwise "internal" series $\sum_{s=p+1}^{\infty}\frac{a_{-s}}{m^s}$ and so forth will turn out to be those which are diverging that it was noted at the end p. 2b.

Further, on (1.353) let us have

$$\frac{1}{\beta-k}\{\eta e^{m\eta}\} = \frac{\eta e^{m\eta}}{m-k} - \frac{e^{m\eta}-e^{k\eta}}{(m-k)^2},$$

$$\boxed{\begin{aligned}
&\frac{1}{\beta-k}\{\eta \operatorname{ch} m\eta\} = \frac{\eta(k \operatorname{ch} m\eta + m \operatorname{sh} m\eta)}{m^2-k^2} - \\
&-\frac{(m^2+k^2)\operatorname{ch} m\eta + 2mk \operatorname{sh} m\eta - (m^2+k^2)e^{k\eta}}{(m^2-k^2)^2}, \\
&\frac{1}{\beta-k}\{\eta \operatorname{sh} m\eta\} = \frac{\eta(k \operatorname{sh} m\eta + m \operatorname{ch} m\eta)}{m^2-k^2} - \\
&-\frac{(m^2+k^2)\operatorname{sh} m\eta + 2mk \operatorname{ch} m\eta - 2mke^{k\eta}}{(m^2-k^2)^2}, \\
&\frac{1}{\beta-k}\{\eta \cos m\eta\} = -\frac{\eta(k \cos m\eta - m \sin m\eta)}{m^2+k^2} + \\
&+\frac{(m^2-k^2)\cos m\eta + 2mk \sin m\eta - (m^2-k^2)e^{k\eta}}{(m^2+k^2)^2}, \\
&\frac{1}{\beta-k}\{\eta \sin m\eta\} = -\frac{\eta(k \sin m\eta + m \cos m\eta)}{m^2+k^2} + \\
&+\frac{(m^2-k^2)\sin m\eta - 2mk \cos m\eta + 2mke^{k\eta}}{(m^2+k^2)^2}. \\
&\qquad\qquad\qquad\qquad m \neq k.
\end{aligned}} \quad (1.364)$$

Here, as in (1.363), the second and third formulas are obtained from the first (by replacement m on-m, and then by addition and subtraction), the fourth and the fifth - by replacement m by im.

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c) The operators $\frac{1}{\beta-k}$ and $\frac{1}{\beta+k}$ are singular and, therefore, permutable. Therefore, applying (1.353), we find

$$\frac{1}{\beta^2-k^2}\varphi(\eta)=\frac{1}{\beta-k}\left\{\frac{1}{\beta+k}\varphi\right\}=\frac{1}{\beta+k}\left\{\frac{1}{\beta-k}\varphi\right\}=e^{-k\eta}\int_0^\eta e^{2k\zeta}d\zeta\int_0^\zeta e^{-k\zeta_1}\varphi(\zeta_1)\,d\zeta_1, \quad (1.365)$$

either, changing the order of integration,

$$\frac{1}{\beta^2-k^2}\varphi(\eta)=e^{k\eta}\int_0^\eta e^{k\zeta_1}\varphi(\zeta_1)\,d\zeta_1\int_{\zeta_1}^\eta e^{-2k\zeta}d\zeta=$$

$$=\frac{1}{2k}\int_0^\eta e^{k(\eta+\zeta_1)}\left(e^{-2k\zeta_1}-e^{-2k\eta}\right)\varphi(\zeta_1)\,d\zeta_1, \quad (1.366)$$

or, finally (after replacing designations ζ_1 by ζ),

$$\boxed{\frac{1}{\beta^2-k^2}\varphi(\eta)=\frac{1}{k}\int_0^\eta \operatorname{sh} k(\eta-\zeta)\,\varphi(\zeta)\,d\zeta.} \quad (1.367)$$

Hence

$$\boxed{\frac{1}{\beta^2+k^2}\varphi(\eta)=\frac{1}{k}\int_0^\eta \sin k(\eta-\zeta)\varphi(\zeta)d\zeta.} \quad (1.368)$$

d) After presenting sine in complex form and after using (1.361), it is not difficult to compute the following integrals:

$$\int_0^x \sin k(x-t)\,dt = \frac{1}{k}(1-\cos kx),$$

$$\int_0^x t\sin k(x-t)\,dt = \frac{1}{k^2}(kx-\sin kx),$$

$$\int_0^x t^n \sin k(x-t)\,dt = \frac{n!}{k^{n+1}}\left[\sum_{p=0}^{n}\left(\frac{1+(-1)^{n+p}}{2}(-1)^{\frac{n-p}{2}}\frac{(kx)^p}{p!}\right) - \right.$$
$$\left. -(-1)^{\frac{n}{2}}\frac{1+(-1)^n}{2}\cos kx - (-1)^{\frac{n-1}{2}}\frac{1-(-1)^n}{2}\sin kx\right]. \quad (1.369)$$

Then from (1.368) we obtain

$$\frac{1}{\beta^2+k^2}\{1\} = \frac{1}{k^2}(1-\cos k\eta),\quad \frac{1}{\beta^2+k^2}\{\eta\} = \frac{1}{k^3}(k\eta-\sin k\eta),$$
$$\frac{1}{\beta^2+k^2}\{\eta^n\} = \frac{n!}{k^{n+2}}\left[\sum_{p=0}^{n}\frac{1+(-1)^{n+p}}{2}(-1)^{\frac{n-2}{2}}\frac{(k\eta)^p}{p!} - \right. \quad (1.370)$$
$$\left. -(-1)^{\frac{n}{2}}\frac{1+(-1)^n}{2}\cos k\eta - (-1)^{\frac{n-1}{2}}\frac{1-(-1)^n}{2}\sin k\eta\right].$$

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e) Directly on (1.368) we find

$$\frac{1}{\beta^2+k^2}\{e^{m\eta}\} = \frac{ke^{m\eta}-m\sin k\eta-k\cos k\eta}{k(m^2+k^2)},$$
$$\frac{1}{\beta^2+k^2}\{\mathrm{ch}\,m\eta\} = \frac{\cos m\eta-\cos k\eta}{m^2+k^2},$$
$$\frac{1}{\beta^2+k^2}\{\mathrm{sh}\,m\eta\} = \frac{k\,\mathrm{sh}\,m\eta-m\sin k\eta}{k(m^2+k^2)}, \quad (1.371)$$
$$\frac{1}{\beta^2+k^2}\{\cos m\eta\} = \frac{\cos m\eta-\cos k\eta}{k^2-m^2},$$
$$\frac{1}{\beta^2+k^2}\{\sin m\eta\} = \frac{k\sin m\eta-m\sin k\eta}{k(k^2-m^2)},$$
$$m \neq k.$$

and further,

$$
\begin{aligned}
\frac{1}{\beta^2+k^2}\{\eta e^{m\eta}\} &= \frac{\eta e^{m\eta}}{m^2+k^2} - \frac{2me^{m\eta}}{(m^2+k^2)^2} + \\
&\quad + \frac{m^2-k^2}{k(m^2+k^2)^2}\sin k\eta + \frac{2m\cos k\eta}{(m^2+k^2)^2}, \\
\frac{1}{\beta^2+k^2}\{\eta \operatorname{ch} m\eta\} &= \frac{\eta \operatorname{ch} m\eta}{m^2+k^2} - \frac{2m \operatorname{sh} m\eta}{(m^2+k^2)^2} + \\
&\quad + \frac{m^2-k^2}{k(m^2+k^2)^2}\sin k\eta, \\
\frac{1}{\beta^2+k^2}\{\eta \operatorname{sh} m\eta\} &= \frac{\eta \operatorname{sh} m\eta}{m^2+k^2} - \frac{2m \operatorname{ch} m\eta}{(m^2+k^2)^2} + \\
&\quad + \frac{2m\cos k\eta}{(m^2+k^2)^2}, \\
\frac{1}{\beta^2+k^2}\{\eta \cos m\eta\} &= \frac{\eta \cos m\eta}{k^2-m^2} + \\
&\quad + \frac{2m\sin m\eta}{(k^2-m^2)^2} - \frac{m^2+k^2}{k(k^2-m^2)^2}\sin k\eta, \\
\frac{1}{\beta^2+k^2}\{\eta \sin m\eta\} &= \frac{\eta \sin m\eta}{k^2-m^2} - \frac{2m\cos m\eta}{(k^2-m^2)^2} + \\
&\quad + \frac{2m\cos k\eta}{(k^2-m^2)^2}, \qquad m \neq k.
\end{aligned}
\tag{1.372}
$$

The observation, which concerns formulas (1.363), in equal measure is related also to formulas (1.371).

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f) Keeping in mind the interchangeability of singular operators (1.367) and (1.368), we find

$$\frac{1}{\beta^4-k^4}\varphi(\eta)=\frac{1}{\beta^2-k^2}\left\{\frac{1}{\beta^2+k^2}\varphi(\eta)\right\}=\frac{1}{k^3}\int_0^{\eta}\operatorname{sh}k(\eta-\zeta)\,d\zeta\int_0^{\zeta}\sin k(\zeta-\zeta_1)\,\varphi(\zeta_1)\,d\zeta_1. \quad (1.373)$$

By changing the order of integration, by producing elementary conversions and being returned in the designation of the variable of integration from $\zeta_1\kappa\zeta$, we will obtain finally

$$\frac{1}{\beta^4-k^4}\varphi(\eta)=\frac{1}{2k^3}\int_0^{\eta}[\operatorname{sh}k(\eta-\zeta)-\sin k(\eta-\zeta)]\,\varphi(\zeta)\,d\zeta. \quad (1.374)$$

After replacing here k (on) $k\sqrt{i}=\frac{1+i}{\sqrt{2}}k$, we will obtain

$$\boxed{\frac{1}{\beta_4+k_4}\varphi(\eta)=\frac{1}{k^3\sqrt{2}}\int_0^{\eta}\left[\sin\frac{k(\eta-\zeta)}{\sqrt{2}}\operatorname{ch}\frac{k(\eta-\zeta)}{\sqrt{2}}-\cos\frac{k(\eta-\zeta)}{\sqrt{2}}\operatorname{sh}\frac{k(\eta-\zeta)}{\sqrt{2}}\right]\varphi(\zeta)\,d\zeta.} \quad (1.375)$$

g) In a number of cases it is convenient to apply formula (1.356). So, using (1.356) (here k = 1) and the first of formulas (1.372) (here k = $\sqrt{\frac{2}{5}}$, $m=1$). let us find

$$\frac{1}{2+5(\beta+1)^2}\{\eta\}=e^{-\eta}\cdot\frac{1}{2+5\beta^2}\{\eta e^{\eta}\}=\frac{e^{-\eta}}{5}\cdot\frac{1}{\beta^2+\frac{2}{5}}\{\eta e^{\eta}\}=$$

$$=\frac{e^{-\eta}}{5}\left\{\frac{\eta e^{-\eta}}{1+\frac{2}{5}}-\frac{2e^{\eta}}{\left(1+\frac{2}{5}\right)^2}+\frac{1-\frac{2}{5}}{\sqrt{\frac{2}{5}}\left(1+\frac{2}{5}\right)^2}\sin\eta\sqrt{\frac{2}{5}}+\right.$$

$$+\frac{2}{\left(1+\frac{2}{5}\right)^2}\cos\eta\sqrt{\frac{2}{5}}\Bigg| = \frac{1}{7}\eta-\frac{10}{49}+\frac{3\sqrt{5e^{-\eta}}}{49\sqrt{2}}\sin\eta\sqrt{\frac{2}{5}}+$$
$$+\frac{10e^{-\eta}}{49}\cos\eta\sqrt{\frac{2}{5}}. \qquad (1.376)$$

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h) It is sometimes convenient to use and (1.359). So, using (1.359) and the latter from formulas (1.371) and (1.372) (here m = 1, k = i), we will obtain

$$\frac{2\beta}{(1-\beta^2)^2}\sin\eta = \left(\frac{\partial}{\partial\beta}\frac{1}{1-\beta^2}\right)\{\sin\eta\} = \frac{1}{1-\beta_2}\{\eta\sin\eta\} -$$
$$-\eta\frac{1}{1-\beta^2}\{\sin\eta\} = -\frac{\eta\sin\eta}{i^2-1}+\frac{2\cos\eta}{(i^2-1)^2}-\frac{2\cos i\eta}{(i^2-1)^2}+ \quad (1.377)$$
$$+\eta\frac{i\sin\eta-\sin i\eta}{i(i^2-1)} = \frac{1}{2}(\cos\eta-\operatorname{ch}\eta+\eta\operatorname{sh}\eta).$$

Here, of course, it was possible to find operator's value above function sine η directly from formula (1.346). However,

calculations would turn out to be bulkier, since it was necessary to expand operator in a series according to degrees β.

i) Let us give an example of the use of integration of operator-function for operator's realization. From first formula (1.363)

$$\frac{1}{\beta-\xi}e^{m\eta}=\frac{e^{m\eta}-e^{\xi\eta}}{m-\xi}. \tag{1.378}$$

Here functions $a_{-s}(\xi)=\xi^{s-1}$ are integrated, and their integrals are evenly continuous in $0 \leqslant \xi \leqslant 1$, which means, possibly the application/use of formula (1.313). Therefore there exists

$$\left(\int_0^1 \frac{d\xi}{\beta-\xi}\right)\{e^{m\eta}\}=\int_0^1 \frac{e^{m\eta}-e^{\xi\eta}}{m-\xi}d\xi. \tag{1.379}$$

After fulfilling integration, we will obtain

$$\ln\left(1-\frac{1}{\beta}\right)\{e^{m\eta}\}=\left(-\frac{1}{\beta}-\frac{1}{2\beta^2}-\frac{1}{3\beta^3}-\ldots\right)e^{m\eta}=$$
$$=e^{m\eta}\left\{\ln\frac{m-1}{m}-Ei(-m\eta)+Ei[-(m-1)\eta]\right\}. \tag{1.380}$$

j) Analogously is utilized differentiation of operator-function. So, without resorting to (1.351), but taking into account the differentiability of functions $a_{-s}(\xi)$ from the preceding/previous example and differentiating (1.378) with respect to ξ (on the basis (1.315)), we find:

$$\frac{1}{(\beta-\xi)^2}e^{m\eta}=\frac{e^{m\eta}-[1+(m-\xi)\eta]e^{\xi\eta}}{(m-\xi)^2}. \tag{1.381}$$

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k) If singular operator is expressed by a complex proper fraction of type (1.301), then a series of its realization is most simple decomposed fraction on the simplest (basis/base: p. g §7) and then used (1.265), (1.351), (1.353), (1.368) and so forth. for example,

$$\frac{6\beta^2-\beta+1}{\beta^3-\beta}=-\frac{1}{\beta}+\frac{3}{\beta-1}+\frac{4}{\beta+1}. \qquad (1.382)$$

Therefore, using (1.265) and (1.363), we determine

$$\frac{6\beta^2-\beta+1}{\beta^3-\beta}\{e^{2\eta}\}=-\frac{e^{2\eta}-1}{2}+3\frac{e^{2\eta}-e^{\eta}}{2-1}+4\frac{e^{2\eta}-e^{-\eta}}{2-(-1)}=$$
$$=\frac{1}{2}-\frac{4}{3}e^{-\eta}-3e^{\eta}+\frac{23}{6}e^{2\eta}. \qquad (1.383)$$

l) **In conclusion** let us consider irrational operator $\frac{1}{\sqrt{\beta^2-k^2}}$. Function $\frac{1}{\sqrt{z^2-k^2}}$ has the singular points $z = \pm$ ki and is **regular** in the **vicinity of** the infinitely receded point. Therefore the corresponding to it operator is singular. Consequently, occurs the **expansion**

$$\frac{1}{\sqrt{\beta^2-k^2}}=\frac{1}{\beta}\left(1-\frac{k^2}{\beta^2}\right)^{-\frac{1}{2}}=\frac{1}{\beta}\left(1+\frac{1}{2}\frac{k^2}{\beta^2}+\frac{1\cdot 3}{2\cdot 4}\frac{k^4}{\beta^4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{k^6}{\beta^6}+\right.$$
$$\left.+\dots\right)=\frac{1}{\beta}+\sum_{s=1}^{\infty}\frac{(2s-1)!!}{(2s)!!}\frac{k^{2s}}{\beta^{2s+1}}. \qquad (1.384)$$

But

$$(2s)!! = 2^s \cdot s!, \quad (2s-1)!! = \frac{(2s)!!(2s-1)!!}{(2s)!!} = \\ = \frac{(2s)!}{(2s)!!} = \frac{(2s)!}{2^s \cdot s!} \quad (1.385)$$

and, which means,

$$a_{-1} = 1, \quad a_{-2s} = 0, \quad a_{-(2s+1)} = \frac{k^{2s} \cdot (2s)!}{2^{2s} \cdot (s!)^2}, \quad s = 0, 1, 2, \ldots \quad (1.386)$$

Then

$$\sum_{s=1}^{\infty} \frac{a_{-s}(\eta-\zeta)^{s-1}}{(s-1)!} = \sum_{s=0}^{\infty} \frac{k^{2s}(2s)!(\eta-\zeta)^{2s}}{2^{2s}(2s)!(s!)^2} = \sum_{s=0}^{\infty} \frac{[k(\eta-\zeta)]^{2s}}{2^{2s} \cdot (s!)^2} = \\ = I_0[k(\eta-\zeta)], \quad (1.387)$$

where I_0 is a Bessel function of apparent/imaginary argument. The last/latter expression is written on the basis of formula (6.457) from [37].

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Substituting (1.387) in (1.279), we obtain

$$\boxed{\frac{1}{\sqrt{\beta^2-k^2}} \varphi(\eta) = \int_0^{\eta} \varphi(\zeta) I_0[\eta(\eta-\zeta)]\, d\zeta = \int_0^{\eta} \varphi(\eta-\zeta) I_0(k\zeta)\, d\zeta.} \quad (1.388)$$

Substituting k on ik and taking into account that (see (6.407) from [37])

$$I_0(ix) = J(-x) = J_0(x). \quad (1.389)$$

we convert (1.388) to

$$\boxed{\frac{1}{\sqrt{\beta^2+k^2}} \varphi(\eta) = \int_0^{\eta} \varphi(\zeta) J_0[k(\eta-\zeta)]\, d\zeta = \int_0^{\eta} \varphi(\eta-\zeta) J_0(k\zeta) d\zeta.} \quad (1.390)$$

On this let us finish the presentation of the theory of the regular and singular operators. The thecry of the mixed operators, which synthesizes the results of the investigation of both forms of the operators, will be examined in the following chapter.

Chapter 2.

SOME QUESTIONS OF THEORY. Mixed operators.

In chapter 1 were more or less examined in detail regular and singular operators. On the basis of the given there theoretical substantiation, here we will consider the mixed operators, and also some common/general/total questions, connected with the operators.

§9. Mixed operators.

1. Basic determinations.

Mixed operator's concept was introduced in p. *la* §7. General view of this operator

$$L(\beta)=\ldots+\frac{a_{-2}}{\beta^2}+\frac{a_{-1}}{\beta}+a_0+a_1\beta+a_2\beta^2+\ldots=\sum_{s=-\infty}^{\infty}a_s\beta^s, \quad (2.1)$$

or, in the case of the mixed operator-function,

$$L(\xi,\beta)=\ldots+\frac{a_{-2}(\xi)}{\beta^2}+\frac{a_{-1}(\xi)}{\beta}+a_0(\xi)+a_1(\xi)\beta+\ldots=$$
$$=\sum_{s=-\infty}^{\infty}a_s(\xi)\beta^s. \quad (2.2)$$

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To such operators correspond to function L (z) or L (ξ, z), representable in the vicinity of the infinitely receded point by Laurent series

$$L(\beta)\longleftrightarrow L(z)=\ldots+\frac{a_{-1}}{z}+a_0+a_1z+\ldots=\sum_{s=-\infty}^{\infty}a_sz^s,\quad (2.3)$$

$$L(\xi,\beta)\longleftrightarrow L(\xi,z)=\ldots+\frac{a_{-1}(\xi)}{z}+a_0(\xi)+a_1(\xi)z+\ldots=$$

$$=\sum_{s=-\infty}^{\infty}a_s(\xi)z^s.\quad (2.4)$$

It is obvious, function L (z) or L (ξ, z), $\xi\in\Gamma_\xi$ (page 77), it must have the isolated/insulated singular points at the final distance and in z = ∞, the latter cannot be branch point.

In accordance with (1.88) and (1.258)

$$L(\beta)=L^+(\beta)+L^-(\beta).\quad (2.5)$$

Then, according to (1.113) and (1.279), where mixed operator's values let us have

$$\boxed{\Phi(\eta)=L(\beta)\varphi(\eta)=\sum_{s=0}^{\infty}a_s\varphi^{(s)}(\eta)+\int_0^{\eta}\varphi(\zeta)\sum_{s=1}^{\infty}\frac{a_{-s}(\eta-\zeta)^{s-1}}{(s-1)!}d\zeta.}\quad (2.6)$$

and for the value mixed operator-function

$$\Phi(\xi,\eta)=L(\xi,\beta)\varphi(\eta)=\sum_{s=0}^{\infty}a_s(\xi)\varphi^{(s)}(\eta)+\int_0^{\eta}\varphi(\zeta)\sum_{s=1}^{\infty}\frac{a_{-s}(\xi)(\eta-\zeta)^{s-1}}{(s-1)!}d\zeta. \tag{2.7}$$

As concerns domain ω_L the determination of the mixed operator, on the basis p. 2 § 5 and p. 1 § 7 it is possible to indicate several versions:

a) is infinitely much $a_s\neq 0$ with s $\geqslant$ 0 and s < 0, and domain Γ (page 39) is infinite; then

$$\omega_L=\omega^{(A)}\cdot\omega_{L-1}, \tag{2.8}$$

i.e. $\varphi(\eta)$ must be that which is integrated in Γ and satisfy condition (1.91). [Page 95]

b) is infinitely much $a_s\neq 0$ with s $\geqslant$ 0 and s < 0, and domain is final; then

$$\omega_L=\omega^{(A)}, \tag{2.9}$$

i.e. $\varphi(\eta)$ it must satisfy condition (1.91);

c) is by n of the coefficients $a_s\neq 0$ with $s>0$ and infinitely much $a_s\neq 0$ with $s<0$, but domain Γ is infinite; then (see page 39)

$$\omega_L=\omega_{Ln}\cdot\omega_{L-1}, \tag{2.10}$$

i.e. $\varphi(\eta)$ it must be that integrated and n once that which is differentiated in Γ and so forth.

If $\varphi \in \omega_L$, then series in (2.6) and (2.7) converge absolutely and evenly.

The interchangeable operators can be algebraic rational, irrational, transcendental, for example,

$$\frac{\beta^2+2}{\beta^4+3\beta},\ \sqrt{\beta^2-1},\ e^{\sqrt{\beta^2-k^2}}. \tag{2.10a}$$

In the case of the mixed operator-functions let us assume that always $\xi \in \Gamma_\xi$ (see page 39). Therefore, for example, if $\xi \in [0,1]$, then $\frac{\beta}{1+\xi\beta}$ is not the mixed (but, and also, therefore, by regular or singular) operator. Let us name it special operator. The number special includes also the operators, for whom is impossible the representation in the form of Laurent series in the vicinity of the infinitely receded point, for example

$$\ln\beta,\ \sqrt{\beta-k},\ \frac{1}{\sin\beta},\ \mathrm{tg}\,\beta. \tag{2.11}$$

In the present work special operators are examined only casually.

2. Properties of the mixed operators.

Since the mixed operator is the sum of the regular and singular operators, its properties easily are obtained from the properties, studied in p. p. 3 and 4 § 5 and p. 2 § 7. Special attention they require, however, the properties, connected with the product of the operators, and them we will consider in the following point/items of the present paragraph.

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a) If exist L_1 and L_2, then there exists and $L = L_1 + L_2$, whereupon

$$L\varphi = L_1\varphi + L_2\varphi. \tag{2.12}$$

b) If $\varphi(\eta) \in \omega_L$, that also $\Phi \in \omega_L$.

c) If there exists $L_2\varphi$, there exists and

$$L_1L_2\varphi = L_1\{L_2\varphi\}. \tag{2.13}$$

d) Operator L (β) is linear. It is more that,

$$L\left\{\sum_{s=1}^{\infty} A_s\, \varphi_s(\eta)\right\} = \sum_{s=1}^{\infty} A_s L\{\varphi_s(\eta)\}, \tag{2.14}$$

if in curly braces a series confronting converges evenly and absolutely, and $\varphi \in \omega_L$ with $\eta \in \Gamma$. Hence, in particular,

$$L\{0\} = 0. \tag{2.15}$$

e) Operator L (ξ, β) unlimitedly "we differentiate" with respect to β whereupon on the basis p. 5 §6 and p. 5 §8

$$\boxed{\frac{\partial L(\xi,\beta)}{\partial\beta}\{\varphi(\eta)\} = L(\xi,\beta)\{\eta\varphi(\eta)\} - \eta L(\xi,\beta)\{\varphi(\eta)\}} \quad (2.16)$$

and generally

$$\frac{\partial^n L(\xi,\partial\beta^n)}{\partial\beta^n}\{\varphi(\eta)\} = L(\xi,\beta)\{\eta^n\varphi(\eta)\} - n\eta L(\xi,\beta)\{\eta^{n-1}\varphi(\eta)\} + \\ + \frac{n(n-1)}{2!}\eta^2 L(\xi,\beta)\{\eta^{n-2}\varphi(\eta)\} + \ldots + (-1)^n\eta^n L(\xi,\beta)\{\varphi(\eta)\}. \quad (2.17)$$

Therefore it is possible to speak about "differential equations" for the operators. So, the operator

$$L(\beta) = \frac{1}{1+\beta}\sin\beta + \frac{1}{\beta}\cos\beta + \frac{1}{5}e^{2\beta} \quad (2.18)$$

it satisfies the equation

$$\frac{\partial^2 L}{\partial\beta^2} + L = e^{2\beta}. \quad (2.19)$$

f) If with $s>0$ coefficients $a_s(\xi)$ satisfy conditions (1.150), and with $s<0$ are satisfied the conditions, indicated in p. 2k §7, then operator-function L (ξ, β) is differentiated and integrated by ξ, for example,

$$\frac{d}{d\xi}(\beta^2-\xi^2)^{3/2} = -3\xi\sqrt{\beta^2-\xi^2}, \quad \int_0^1 \cos\frac{\xi}{\beta}\,d\xi = \beta\sin\frac{1}{\beta}. \quad (2.20)$$

It is possible to examine differential equations for operator-functions.

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So, the solution to the equation

$$\frac{d^2L}{d\xi^2}+\beta^2L=1 \qquad (2.21)$$

under the conditions

$$L(0,\beta)=1,\quad \frac{dL(0,\beta)}{d\xi}=1 \qquad (2.22)$$

it will be

$$L(\xi,\beta)=\frac{1}{\beta}\sin\xi\beta+\frac{1-\beta^2}{\beta^2}\cos\xi\beta+\frac{1}{\beta^2}{}^{*}. \qquad (2.23)$$

FOOTNOTE [1]. For treatment of fractions see below (p. 6). ENDFOOTNOTE.

g) from p. 4 §6 and p. 4 § 8 follows that

$$\boxed{L(\xi,\beta+k)\varphi(\eta)=e^{-k\eta}L(\xi,\beta)\{e^{k\eta}\varphi(\eta)\}.} \qquad (2.24)$$

h) Occurs the following similarity transformation: if

$$L(\xi,\beta)\varphi(\eta)=\Phi(\xi,\eta), \qquad (2.25)$$

that

$$\boxed{L(\xi,m\beta)\varphi\left(\frac{\eta}{m}\right)=\Phi\left(\xi,\frac{\eta}{m}\right).} \qquad (2.26)$$

In fact, let, as usual,

$$\left.\begin{aligned}
&L^+(\xi,\beta)\varphi(\eta)=\sum_{s=0}^{\infty}a_s(\xi)\beta^s\varphi(\eta)=\sum_{s=0}^{\infty}a_s(\xi)\frac{d^s\varphi(\eta)}{d\eta^s}=\\
&=\Phi^+(\xi,\eta),\\
&L^-(\xi,\beta)\varphi(\eta)=\sum_{s=1}^{\infty}\frac{a_{-s}(\xi)}{\beta^s}\varphi(\eta)=\\
&=\int_0^{\eta}\varphi(\zeta)\sum_{s=1}^{\infty}\frac{a_{-s}(\xi)(\eta-\zeta)^{s-1}}{(s-1)!}d\zeta=\Phi^-(\xi,\eta),\\
&L(\xi,\beta)\varphi(\eta)=[L^+(\xi,\beta)+L^-(\xi,\beta)]\varphi(\eta)=\\
&=\Phi^+(\xi,\eta)+\Phi^-(\xi,\eta)=\Phi(\xi\eta).
\end{aligned}\right\}\qquad(2.27)$$

Hence

$$L^+(\xi,m\beta)\varphi\left(\frac{\eta}{m}\right)=\sum_{s=0}^{\infty}a_s(\xi)m^s\beta^s\varphi\left(\frac{\eta}{m}\right)=$$

$$=\sum_{s=0}^{\infty}a_s(\xi)m^s\frac{d^s\varphi\left(\frac{\eta}{m}\right)}{d\eta^s}=\sum_{s=0}^{\infty}a_s(\xi)\frac{d^s\varphi\left(\frac{\eta}{m}\right)}{d\left(\frac{\eta}{m}\right)^s},$$

$$L^-(\xi,m\beta)\varphi\left(\frac{\eta}{m}\right)=\sum_{s=1}^{\infty}\frac{a_{-s}(\xi)}{m^s\beta^s}\varphi\left(\frac{\eta}{m}\right)=\int_0^{\eta}\varphi\left(\frac{\zeta}{m}\right)\sum_{s=1}^{\infty}\frac{a_{-s}(\eta-\zeta)^{s-1}}{(s-1)!\,m^s}d\zeta=$$

$$=\int_0^{\eta/m}\varphi(\zeta_1)\sum_{s=1}^{\infty}\frac{a_{-s}\left(\frac{\eta}{m}-\zeta_1\right)^{s-1}}{(s-1)!}d\zeta_1.\qquad(2.28)$$

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Comparing (2.28) with first two formulas (2.27), is detected, that

$$L\pm(\xi, m\beta)\,\varphi\left(\frac{\eta}{m}\right) = \Phi^{\pm}\left(\xi, \frac{\eta}{m}\right). \qquad (2.29)$$

But then from third formula (2.27) it is obtained (2.26). The obtained result can be formulated also as follows: if we in equality (2.25) replace of variables

$$\eta = my, \qquad (2.30)$$

that

$$L(\xi, m\beta)\,\varphi(y) = \Phi(\xi, y). \qquad (2.31)$$

3. Multiplication of the operators.

a) first let us consider the elementary products, in which enter the integral and differentiation operator. Direct calculations, based on determinations (1.2), (1.24) and (1.265), give:

with $0 \leqslant n \leqslant s$

$$\beta^n \cdot \frac{1}{\beta^s}\varphi = \frac{1}{\beta^{s-n}}\varphi = \frac{1}{(s-n-1)!}\int_0^\eta (\eta-\zeta)^{s-n-1}\varphi(\zeta)\,d\zeta,$$

(1) при $n \geqslant s$

$$\beta^n \cdot \frac{1}{\beta^s}\varphi = \beta^{n-s}\varphi = \varphi^{(n-s)}(\eta),$$

(1) при $1 \leqslant n \leqslant s$

$$\frac{1}{\beta^s}\cdot\beta^n\varphi = \frac{1}{\beta^{s-n}}\varphi - \sum_{p=s-n}^{s-1}\varphi^{(p-s+n)}(0)\frac{\eta^p}{p!} =$$

$$= \frac{1}{(s-n-1)!}\int_0^\eta (\eta-\zeta)^{s-n-1}\varphi(\zeta)\,d\zeta - \sum_{p=s-n}^{s-1}\varphi^{(p-s+n)}(0)\frac{\eta^p}{p!},$$

(1) при $n \geqslant s$

$$\frac{1}{\beta^s}\beta^n\varphi = \beta^{n-s}\varphi - \sum_{p=0}^{s-1}\varphi^{(p-s+n)}(0)\frac{\eta^p}{p!} =$$

$$= \varphi^{(n-s)}(\eta) - \sum_{p=0}^{s-1}\varphi^{(p-s+n)}(0)\frac{\eta^p}{p!}. \qquad (2.32)$$

Key: (1). with.

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Hence easily is perceived noted even in (1.266) the noninterchangeability of integral and differential operators.

b) Let us pass further to the more complex operators. Let

$$L_1^+(\beta) = \sum_{s=0}^{\infty} a_s^{(1)}\beta^s, \quad L_2^+(\beta) = \sum_{s=0}^{\infty} a_s^{(2)}\beta^s,$$
$$L_1^-(\beta) = \sum_{s=1}^{\infty} a_{-s}^{(1)}\beta^{-s}, \quad L_2^-(\beta) = \sum_{s=1}^{\infty} a_{-s}^{(2)}\beta^{-s}. \qquad (2,33)$$

For the product of the regular operators according to (1.122) let us have

$$L_1^+(\beta)\,L_2^+(\beta)\,\varphi = \sum_{s=0}^{\infty} \sum_{p=0}^{s} a_p^{(1)} a_{s-p}^{(2)} \Big) \varphi^{(s)}(\eta) = \sum_{s=0}^{\infty} \left(\sum_{n=0}^{s} a_n^{(2)} a_{s-n}^{(1)} \right) \varphi^{(s)}(\eta). \quad (2.34)$$

while for the singular operators according to (1.296)

$$L_1^-(\beta)\,L_2^-(\beta)\,\varphi = \int_0^{\eta} \varphi(\zeta) \sum_{s=2}^{\infty} \left(\sum_{p=1}^{s-1} a_{-p}^{(1)} a_{-(s-p)}^{(2)} \right) \frac{(\eta-\zeta)^{s-1}}{(s-1)!}\, d\zeta =$$

$$= \int_0^{\eta} \varphi(\zeta) \sum_{s=2}^{\infty} \left(\sum_{n=1}^{s-1} a_{-n}^{(2)} a_{-(s-n)}^{(1)} \right) \frac{(\eta-\zeta)^{s-1}}{(s-1)!}\, d\zeta. \quad (2.35)$$

These formulas can be written in symmetrical relative to $a^{(1)}$ and $a^{(2)}$ form, for example,

$$L_1^+(\beta)\,L_2^+(\beta)\,\varphi = \sum_{s=0}^{\infty} \left(\sum_{p=0}^{s} \frac{a_p^{(1)} a_{s-p}^{(2)} + a_p^{(2)} a_{s-p}^{(1)}}{2} \right) \frac{(\eta-\zeta)^{s-1}}{(s-1)!}\, d\zeta. \quad (2.36)$$

Is hence visible established/installed into §5 and 7 interchangeability of the products of the regular and singular operators.

c) Now, being based on the absolute and uniform convergence of those being encountered subsequently together (see §5 and 7), and also on the linearity of the operators L^+ and L^- and using formulas (2.32), let us consider regular operator's product and singular and singular by regular. We have

$$L_1^+(\beta)\left\{\frac{1}{\beta^s}\varphi\right\} = \sum_{n=0}^{\infty} a_n^{(1)}\beta^n\left\{\frac{1}{\beta^s}\varphi\right\} = \sum_{n=0}^{s-1}\frac{a_n^{(1)}}{(s-n-1)!}\int_0^\eta(\eta-\zeta)^{s-n-1}\varphi(\zeta)d\zeta + \\ + \sum_{n=s}^{\infty} a_n^{(1)}\varphi^{(n-s)}(\eta). \tag{2.37}$$

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Then

$$L_1^+(\beta)L_2^-(\beta)\varphi(\eta) = L_1^+\{L_n^-\varphi\} = L_1^+\left\{\sum_{s=1}^{\infty}\frac{a_{-s}^{(2)}}{\beta^s}\varphi\right\} = \\ = \sum_{s=1}^{\infty} a_{-s}^{(2)}L_1^+(\beta)\left\{\frac{1}{\beta^s}\varphi\right\} = \sum_{s=1}^{\infty} a_{-s}^{(2)}\sum_{n=0}^{s-1}\frac{a_n^{(1)}}{(s-n-1)!}\int_0^\eta(\eta-\zeta)^{s-n-1}\varphi(\zeta)\,d\zeta + \\ + \sum_{s=1}^{\infty} a_{-s}^{(2)}\sum_{n=s}^{\infty} a_n^{(1)}\varphi^{(n-s)}(\eta). \tag{2.38}$$

or

$$L_1^+L_2^-\varphi = \int_0^\eta\varphi(\zeta)\sum_{p=1}^{\infty}\left(\sum_{n=0}^{\infty} a_n^{(1)}a_{-(n+p)}^{(2)}\right)\frac{(\eta-\zeta)^{p-1}}{(p-1)!}\,d\zeta + \\ + \sum_{p=0}^{\infty}\left(\sum_{s=1}^{\infty} a_{s+p}^{(1)}a_{-s}^{(2)}\right)\varphi^{(p)}(\eta). \tag{2.39}$$

Further with $n \neq 0$

$$L_1^-(\beta)\{\beta^n\varphi\} = \sum_{s=1}^{n} a_{-s}^{(1)}\varphi^{(n-s)}(\eta) - \sum_{s=1}^{n} a_{-s}^{(1)} \sum_{p=0}^{s-1}\varphi^{(p-s+n)}(0)\frac{\eta^p}{p!} +$$

$$+\int_0^\eta \varphi(\zeta)\sum_{s=n+1}^\infty a_{-s}^{(1)}\frac{(\eta-\zeta)^{s-n-1}}{(s-n-1)!}d\zeta - \sum_{s=n+1}^\infty a_{-s}^{(1)}\sum_{p=s-n}^{s-1}\varphi^{(p-s+n)}(0)\frac{\eta^p}{p!}, \quad (2.40)$$

therefore

$$L_1^- L_2^+\varphi = L_1^-\{L_2^+\varphi\} = L_1^-\left\{\sum_{n=0}^{\infty} a_n^{(2)}\varphi\right\} = \sum_{n=0}^{\infty} a_n^{(2)} L_1^-(\beta)\{\beta^n\varphi\} =$$

$$= a_0^{(2)}\int_0^\eta \varphi(\zeta)\sum_{s=1}^\infty \frac{a_{-s}^{(1)}}{(s-1)!}(\eta-\zeta)^{s-1}d\zeta +$$

$$+\int_0^\eta \varphi(\zeta)\sum_{n=1}^\infty a_n^{(2)}\sum_{s=n+1}^\infty a_{-s}^{(1)}\frac{(\eta-\zeta)^{s-n-1}}{(s-n-1)!}d\zeta -$$

$$-\sum_{n=1}^\infty a_n^{(2)}\sum_{s=1}^n a_{-s}^{(1)}\varphi^{(n-s)}(\eta) - \sum_{n=1}^\infty a_n^{(2)}\sum_{s=1}^n a_{-s}^{(1)}\varphi^{(p-s+n)}(0)\frac{\eta^p}{p!} -$$

$$-\sum_{n=1}^\infty a_n^{(2)}\sum_{s=n+1}^\infty a_{-s}^{(1)}\sum_{p=s-n}^{s-1}\varphi^{(p-s+n)}(0)\frac{\eta^p}{p!}. \quad (2.41)$$

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After elementary, but sufficiently bulky transformations we will cbtain

$$L_1L_2\varphi = \int_0^\eta \varphi(\zeta)\sum_{p=1}^\infty\left(\sum_{n=0}^\infty a_{-(p+n)}^{(1)}a_n^{(2)}\frac{(\eta-\zeta)^{p-1}}{(p-1)!}d\zeta +\right.$$

$$+\sum_{p=0}^\infty\left(\sum_{s=1}^\infty a_{-s}^{(1)}a_{p+s}^{(2)}\right)\varphi^{(p)}(\eta) - \quad (2.42)$$

$$-\sum_{p=0}^\infty\left[\sum_{n=1}^\infty a_n^{(2)}\left(\sum_{\nu=1}^n a_{-(\nu+p)}^{(1)}\varphi^{(n-\nu)}(0)\right)\right]\frac{\eta^p}{p!}.$$

By changing here by places indices 1 and 2 and by comparing with (2.39), let us find:

$$L_2^- L_1^+ \varphi = L_1^+ L_2^- \varphi - \sum_{p=0}^{\infty}\left[\sum_{n=1}^{\infty} a_n^{(1)}\left(\sum_{\nu=1}^{n} a_{-(\nu+p)}^{(2)}\varphi^{(n-\nu)}(0)\right)\right]\frac{\eta^p}{p!}. \quad (2.43)$$

d) By using results p. b and c, let us find another expression for the product of the mixed operators. Let

$$\left.\begin{aligned} L_1(\beta) = L_1^+(\beta) + L_1^-(\beta) = \sum_{n=0}^{\infty} a_n^{(1)}\beta^n + \sum_{s=1}^{\infty}\frac{a_{-s}^{(1)}}{\beta^s}, \\ L_2(\beta) = L_2^+(\beta) + L_2^-(\beta) = \sum_{n=0}^{\infty} a_n^{(2)}\beta^n + \sum_{s=1}^{\infty}\frac{a_{-s}^{(2)}}{\beta^s}. \end{aligned}\right\} \quad (2.44)$$

Then

$$L_1 L_2\varphi = L_1^+ L_2^+\varphi + L_1^- L_2^-\varphi + L_1^+ L_2^-\varphi + L_1^- L_2^+\varphi. \quad (2.45)$$

After substituting here expressions (2.34), (2.35), (2.39), (2.42) and after leading simple lining/calculations, we will obtain

$$\begin{aligned} L_1 L_2\varphi = {} & \sum_{s=0}^{\infty}\left[\sum_{n=0}^{s} a_n^{(1)} a_{s-n}^{(2)} + \sum_{n=1}^{\infty}\left(a_{s+n}^{(1)} a_{-n}^{(2)} + a_{s+n}^{(2)} a_{-n}^{(1)}\right)\right]\varphi^{(s)}(\eta) + \\ & + \int_0^{\eta}\varphi(\zeta)\left\{\sum_{n=0}^{\infty}\left(a_n^{(1)} a_{-(n+1)}^{(2)} + a_n^{(2)} a_{-(n+1)}^{(1)}\right) + \sum_{s=2}^{\infty}\left[\sum_{n=1}^{s-1} a_{-n}^{(1)} a_{-(s-n)}^{(2)} + \right.\right. \\ & \left.\left. + \sum_{n=0}^{\infty}\left(a_n^{(1)} a_{-(s+n)}^{(2)} + a_n^{(2)} a_{-(s+n)}^{(1)}\right)\right]\frac{(\eta-\zeta)^{s-1}}{(s-1)!}\right\} d\zeta - \\ & - \sum_{p=0}^{\infty}\left[\sum_{n=1}^{\infty} a_n^{(2)}\left(\sum_{\nu=1}^{n} a_{-(\nu+p)}^{(1)}\varphi^{(n-\nu)}(0)\right)\right]\frac{\eta^p}{p!}. \end{aligned} \quad (2.46)$$

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4. Interchangeability and the functional properties of product.

a) unlike the product of the regular or singular operators mixed product (where they enter and L_1^+, and L^-) and the product of the mixed operators in the general case are noncommutative. So, formula (2.43) shows that

$$L_2^- L_1^+ = L_1^+ L_2^- \qquad (2.47)$$

in that and only when

$$-\sum_{n=1}^{\infty} a_n^{(1)} \left(\sum_{\nu=1}^{n} a_{-(\nu+p)}^{(2)} \varphi^{(n-\nu)}(0) \right) = 0 \ \text{при}^{(1)} \ p = 0, 1, 2, \ldots, \qquad (2.48)$$

Key: (1). with.

and this is impossible. It means L^+ and L^- never they commutate.

b) If we in (2.46) interchange the position indices 1 and 2 and result to subtract from (2.46), then we will obtain

$$L_2 L_1 \varphi = L_1 L_2 \varphi - -\sum_{p=0}^{\infty} \left\{ \sum_{n=1}^{\infty} \left[\sum_{\nu=1}^{\infty} (a_n^{(1)} a_{-(\nu+p)}^{(2)} - a_n^{(2)} a_{-(\nu+p)}^{(1)}) \varphi^{(n-\nu)}(0) \right] \right\} \frac{\eta^p}{p!}, \qquad (2.49)$$

where L_1 and L_2 are given by formulas (2.44). Hence follows the necessary and sufficient condition of the interchangeability of the

mixed operators:

$$\sum_{n=1}^{\infty}\left[\sum_{\nu=1}^{n}(a_n^{(1)}a_{-(\nu+p)}^{(2)}-a_n^{(2)}a_{-(\nu+p)}^{(1)})\varphi^{(n-\nu)}(0)\right]=0 \quad \text{при } p=0,1,2,\ldots \tag{2.50}$$

Key: (1). with.

This is possible only in three cases:

1) if

$$a_{-s}^{(1)}=a_{-s}^{(2)}\equiv 0, \tag{2.51}$$

i. e., both operators are regular;

2) if

$$a_n^{(1)}=a_n^{(2)}\equiv 0,\ n\neq 0, \tag{2.52}$$

i. e., both operators are singular or take the form

$$\boxed{L=a_0+\frac{a_{-1}}{\beta}+\frac{a_{-2}}{\beta^2}+\frac{a_{-3}}{\beta^3}+\ldots} \tag{2.53}$$

without expanding operator in a series, this form easy to discover, after computing $\lim_{\beta\to\infty}$ L (β) - it it must be equal to constant).

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3) if

$$a_n^{(2)}=\lambda a_n^{(1)},\ a_{-s}^{(2)}=\lambda a_{-s}^{(1)} \text{ при } a_0^{(2)}\gtreqless a_0^{(1)}, \tag{2.54}$$

Key: (1). with.

i. e., the mixed operators are distinguished to constant -

$$L_2 = \lambda L_1 + C. \tag{2.55}$$

Substantial to note that formula (2.50) shows that if the expansion L_1 (or L_2) contains at least according to one positive and negative degree β, then there is no operator L_2 (or L_1), interchangeable with L_2 (or L_1), besides operator (2.55).

Let us give several examples:

$$\left.\begin{aligned}
&\operatorname{ch}\xi\beta\cdot(1+\beta^2)=(1+\beta^2)\operatorname{ch}\xi\beta,\ \operatorname{ch}\xi\beta\cdot\frac{1}{1+\beta^2}\neq\frac{1}{1+\beta^2}\operatorname{ch}\xi\beta,\\
&\frac{1}{\beta^2+1}\cdot\frac{1}{\sqrt{\beta^2-k^2}}=\frac{1}{\sqrt{\beta^2-k^2}}\cdot\frac{1}{\beta^2+1},\ \operatorname{ch}\frac{\xi}{\beta}\cdot\operatorname{ch}\frac{3}{\beta}=\\
&\qquad=\operatorname{ch}\frac{3}{\beta}\cdot\operatorname{ch}\frac{\xi}{\beta},\\
&\left(\beta-1+\frac{1}{\beta}\right)\left(\beta+1-\frac{1}{\beta}\right)\neq\left(\beta+1-\frac{1}{\beta}\right)\left(\beta-1+\frac{1}{\beta}\right),\\
&\left(\beta-1+\frac{1}{\beta}\right)\left(\beta+1+\frac{1}{\beta}\right)=\left(\beta+1+\frac{1}{\beta}\right)\left(\beta-1+\frac{1}{\beta}\right).
\end{aligned}\right\} \tag{2.56}$$

c) Let

$$L_j(z)=\sum_{s=0}^{\infty}a_s^{(j)}z^s+\sum_{s=1}^{\infty}\frac{a_{-s}^{(j)}}{z^s},\quad j=1,2,3, \tag{2.57}$$

whereupon

$$L_1(z)\,L_2(z)=L_3(z) \tag{2.58}$$

and

$$L_j(z)\longleftrightarrow L_j(\beta). \tag{2.59}$$

Let us explain, which dependence between the product L_1 (β) L_2 (β) and L_3 (β) corresponds (2.58). Let us introduce (2.57) in (2.58):

$$L_1(z)\,L_2(z)=\sum_{n=0}^{\infty}\sum_{s=0}^{\infty} a_n^{(1)} a_s^{(2)} z^{n+s}+$$

$$+\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\frac{b_{-n}^{(1)}\,b_{-s}^{(2)}}{z^{n+s}}+\sum_{n=0}^{\infty}\sum_{s=1}^{\infty}\left(a_n^{(1)}\,b_{-s}^{(2)}+a_n^{(2)}\,b_{-s}^{(1)}\right)=$$

$$=L_3(z)=\sum_{s=0}^{\infty} a_s^{(3)} z^s+\sum_{s=1}^{\infty}\frac{a_{-s}^{(3)}}{z^s}. \qquad (2.60)$$

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After some transformations this expression can be given to

$$L_1(z)\,L_2(z)=\sum_{s=0}^{\infty}\left[\sum_{n=0}^{s} a_n^{(1)} a_{s-n}^{(2)}+\sum_{n=1}^{\infty}\left(a_{s+n}^{(1)} a_{-n}^{(2)}+a_{s+n}^{(2)} b_{-n}^{(1)}\right)\right] z^s+$$

$$+\sum_{n=0}^{\infty}\left(a_n^{(1)} a_{-(n+1)}^{(2)}+a_n^{(2)} a_{-(n+1)}^{(1)}\right)\frac{1}{z}+$$

$$+\sum_{s=2}^{\infty}\left[\sum_{n=1}^{s-1} a_{-n}^{(1)} a_{-(s-n)}^{(2)}+\sum_{n=0}^{\infty}\left(a_n^{(1)} a_{-(n+s)}^{(2)}+a_n^{(2)} a_{-(n+s)}^{(1)}\right)\right]\frac{1}{z^s}. \qquad (2.61)$$

Thus,

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$$\left.\begin{aligned} a_s^{(3)} &= \sum_{n=0}^{s} a_n^{(1)} a_{s-n}^{(2)} + \sum_{n=1}^{\infty} (a_{s+n}^{(1)} a_{-n}^{(2)} + a_{s+n}^{(2)} a_{-n}^{(1)}), \; s = 0, 1, 2, \ldots \\ a_{-1}^{(3)} &= \sum_{n=0}^{\infty} (a_n^{(1)} a_{-(n+1)}^{(2)} + a_n^{(2)} a_{-(n+1)}^{(1)}), \\ a_{-s}^{(3)} &= \sum_{n=1}^{s-1} a_{-n}^{(1)} a_{-(p-n)}^{(2)} + \sum_{n=0}^{\infty} (a_n^{(1)} a_{-(n+s)}^{(2)} + a_n^{(2)} a_{-(n-s)}^{(1)}), \; s = 2, 3, \ldots \end{aligned}\right\} \quad (2.62)$$

Then regarding operator and (2.57)

$$L_3(\beta)\,\varphi(\eta) = \sum_{s=0}^{\infty} \left[\sum_{n=0}^{s} a_n^{(1)} a_{s-n}^{(2)} + \sum_{n=1}^{\infty} (a_{s+n}^{(1)} a_{-n}^{(2)} + a_{s+n}^{(2)} a_{-n}^{(1)}) \right] \varphi^{(s)}(\eta) + \\ + \int_0^{\eta} \varphi(\zeta) \sum_{n=0}^{\infty} (a_n^{(1)} a_{-(n+1)}^{(2)} + a_n^{(2)} a_{-(n+1)}^{(1)})\, d\zeta + \int_0^{\eta} \varphi(\zeta) \sum_{s=2}^{\infty} \left[\sum_{n=1}^{s-1} a_{-n}^{(1)} a_{-(s-n)}^{(2)} + \right. \\ \left. + \sum_{n=0}^{\infty} (a_n^{(1)} a_{-(n+s)}^{(2)} + a_n^{(2)} a_{-(n+s)}^{(1)}) \right] \frac{(\eta-\zeta)^{s-1}}{(s-1)!}\, d\zeta. \quad (2.63)$$

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Comparing (2.63) with (2.46), is detected, that

$$L_1 L_2 \varphi = L_3 \varphi - \sum_{p=0}^{\infty} \left[\sum_{n=1}^{\infty} a_n^{(2)} \left(\sum_{\nu=1}^{n} a_{-(\nu+p)}^{(1)} \varphi^{(n-\nu)}(0) \right) \right] \frac{\eta^p}{p!} \quad (2.64)$$

and, which means,

$$L_2 L_1 \varphi = L_3 \varphi - \sum_{p=0}^{\infty} \left[\sum_{n=1}^{\infty} a_n^{(1)} \left(\sum_{\nu=1}^{n} a_{-(\nu+p)}^{(2)} \varphi^{(n-\nu)}(0) \right) \right] \frac{\eta^p}{p!}. \quad (2.65)$$

d) Now it is possible to formulate the following affirmation: if L_p (z) are given by formulas (2.57) and it occurs (2.58), then for

that in order to

$$L_1(\beta)\, L_2(\beta) = L_3(\beta) \tag{2.66}$$

(not only formally, but also in sense p. 1 page 27), it is necessary and sufficient in order that in product either the left operator would be regular (then right any), or the right operator took form (2.53) (then left any).

$$\sum_{p=0}^{\infty}\left[\sum_{n=1}^{\infty} a_n^{(1)}\left(\sum_{\nu=1}^{n} a^{(2)}_{-(\nu+p)}\varphi^{(n-\nu)}(\nu)\right)\right]\frac{\eta^p}{p!}=0. \tag{2.67}$$

Proof of need. Let it occurs (2.66). Then in (2.64) must be

$$\sum_{n=1}^{\infty} a_n^{(2)}\left(\sum_{\nu=1}^{n} a^{(1)}_{-(\nu+p)}\varphi^{(n-\nu)}(0)\right)=0,\quad p=0,\ 1,\ 2,\ \ldots, \tag{2.68}$$

a this possibly only in two cases: either with $a^{(1)}_{-s}=0,\ s=1,\ 2$ (i.e. the left operator regular), or with $a^{(2)}_{n}=0,\ n=1,\ 2,\ \ldots$ (i.e. the right operator takes the form (2.53)).

Proof of sufficiency. If in product L_1L_2 the left operator regular (i.e. $a^{(1)}_{-s}=0,\ s=1,\ 2,\ \ldots$) or the right operator takes the form (2.53) (i.e. $a^{(2)}_{n}=0,\ n=1,\ 2,\ \ldots$), that is correct (2.68). Then from (2.64) it follows (2.66).

Consequence 1. If both operators regular or both operators are singular, then under condition (2.58) it is always correct (2.66).

Consequence 2. Any operational equality will not be broken, if

both parts of it are multiplied to the left by regular operator or to the right by the operator of form (2.53).

Observation 1. From (2.66) it does not follow that $L_1 L_2 = L_2 L_1$. For this must be carried out supplementary conditions (p. b).

Observation 2. Everything said is related to the action of the operators above the arbitrary $\varphi \in \omega_Z$.

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For some sets $\varphi(\eta)$ the condition of correctness (2.66) they are expanded:

if

$$\varphi(0) = \varphi'(0) = \ldots = \varphi^{(n_0)}(0) = 0, \qquad (2.69)$$

that with the arbitrary L_1 must be

$$a_s^{(2)} \equiv 0, \ s = n+2, \ n+3, \ldots. \qquad (2.70)$$

This is easy to see from expression (2.68), converting it preliminarily to the form

$$\sum_{n=1}^{\infty} a_n^{(2)} \left(\sum_{s=0}^{n-1} a^{(1)}_{-(n-s+p)} \varphi^{(s)}(0) \right) = 0, \ p = 0, 1, 2, \ldots. \qquad (2.71)$$

Thus, if are satisfied conditions (2.67), then operational product (2.66) possesses the same functional properties as function

L_3 (z) from (2.58).

Let us give several examples:

$$\left.\begin{array}{c}\left(1+\frac{1}{\beta^n}\right)\left(1-\frac{1}{\beta^n}\right)=1-\frac{1}{\beta^{2n}},\\ (\beta+1)\cdot\frac{1}{\beta^2-1}=\frac{1}{\beta-1},\ \sqrt{\beta^2-k^2}\cdot\frac{1}{\beta^2-k^2}=\frac{1}{\sqrt{\beta^2-k^2}},\\ \operatorname{sh}\frac{\xi}{\beta}\operatorname{ch}\frac{\xi}{\beta}=\operatorname{ch}\frac{\xi}{\beta}\cdot\operatorname{sh}\frac{\xi}{\beta}=\frac{1}{2}\operatorname{sh}\frac{2\xi}{\beta},\\ \frac{1}{\beta-1}(\beta^2-1)\neq\beta+1,\ e^{\frac{\xi}{\beta}}\cdot e^{\xi\beta}\neq e^{\xi\left(\frac{1}{\beta}+\beta\right)}.\end{array}\right\} \quad (2.72)$$

It is not difficult to demonstrate also the reverse/inverse affirmation: if correctly (2.66), then it occurs (2.58). For this is sufficient to consider (2.64), (2.46), (2.57), (2.63) and (2.62).

5. On inverse operators.

a) Let $L_1\beta$ be certain cperator. Let us assume that there is operator L_2 (β) such that

$$L_2(\beta)\,L_1(\beta)\,\varphi=\varphi. \quad (2.73)$$

then in accordance with p. 4d,

$$L_2(z)\,L_1(z)=1,\ L_2(z)=\frac{1}{L_1(z)}. \quad (2.74)$$

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where $L_2(z) \longleftrightarrow L_2\beta$, $L_1(z) \longleftrightarrow L_1\beta$.

According to (2.73) and (1.25) operator L_2 will be reverse/inverse for L_1.

Thus, establish/installed that if for any L (β) there is an inverse operator, then it compulsorily takes the form $(L(\beta))^{-1} = \frac{1}{L(\beta)}$:

$$L^{-}(\beta)L(\beta)\varphi = \varphi \qquad (2.75)$$

and

$$L^{-1}(\beta) = \frac{1}{L(\beta)}. \qquad (2.76)$$

b) formula (2.75) and affirmation (2.67) give answer/response to the question concerning that, for which L (β) there are inverse operators. Specifically,: for L (β) there is an inverse operator in that and only that case, when either L (β) has form (2.53), or formal expression $\frac{1}{L(\beta)}$ it is regular operator. Therefore, for example, for the operators

$$\frac{1}{\beta}, \quad \frac{1}{\beta^2-k^2}, \quad \frac{\beta-1}{\beta+1}, \quad \frac{1}{\sqrt{\beta^2-k^2}}, \quad e^{i\xi\beta}, \quad \frac{e^{i\xi\beta}}{1+\xi\beta} \qquad (2.77)$$

reverse/inverse will be respectively

$$\beta, \quad \beta^2-k^2, \quad \frac{\beta+1}{\beta-1}, \quad \sqrt{\beta^2-k^2}, \quad e^{-i\xi\beta}, \quad (1+\xi\beta)e^{-i\xi\beta}, \qquad (2.78)$$

and the operators

$$\beta, \quad \beta^2-k^2, \quad \frac{\beta^2+1}{\beta-1}, \quad \sqrt{\beta^2-k^2}, \quad \beta e^{i\xi\beta}, \quad \sin\xi\beta \qquad (2.79)$$

they do not have reverse/inverse.

6. Fractional operators (operational fractions).

a) First let us examine the only rational fractions. Let us introduce designation

$$P_n(\beta) = \beta^n + A_{n-1}\beta^{n-1} + A_{n-2}\beta^{n-2} + \ldots + A_1\beta + A_0, \quad (2.80)$$

where $n \geqslant 0$ - integer, and A_j - the number (generally speaking complex) or of function of ξ. According to the determinations of regular and singular operator always there are the operators $P_n(\beta)$ and $\frac{1}{P_m(\beta)}$, while according to (2.67), always there are their products, whereupon

$$P_n(\beta) \cdot \frac{1}{P_m(\beta)} = \frac{P_n(\beta)}{P_m(\beta)}. \quad (2.81)$$

This equality can be considered the determination of operational fraction.

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Using this determination and results p. 4 g even 5; it is possible to show that the operational fractions possess the properties of usual fractions (exception is the multiplication of fractions).

1. Fractions are equal to:

$$\frac{P_{n_1}(\beta)}{P_{m_1}(\beta)} = \frac{P_{n2}(\beta)}{P_{m_2}(\beta)} \tag{2.82}$$

in that and only that case, if

$$P_{n_1}(\beta)\,P_{m_2}(\beta) = P_{n_2}(\beta)\,P_{m_1}(\beta). \tag{2.83}$$

2. Denominator is decompose/expanded into the factors:

$$\frac{1}{P_m(\beta)} = \frac{1}{(\beta - k_1)(\beta - k_2)\ldots(\beta - k_m)}, \tag{2.84}$$

where k_s — the roots of polynomial $P_m(z)$.

3. Are valid equalities

$$\frac{\beta - k_1}{\beta - k_1} = \frac{(\beta - k_1)(\beta - k_2)}{(\beta - k_1)(\beta - k_2)} = \ldots = \frac{(\beta - k_1)(\beta - k_2)\ldots(\beta - k_n)}{(\beta - k_1)(\beta - k_2)\ldots(\beta - k_n)} = 1. \tag{2.85}$$

4. Fractional operator will not be changed, if numerator and denominator is multiplied by one and the same polynomial or is shortened fraction by it

$$\frac{P_n(\beta)}{P_m(\beta)} = \frac{(\beta-k)P_n(\beta)}{(\beta-k)P_m(\beta)} = \frac{P_{n_1}(\beta)P_n(\beta)}{P_{n_1}(\beta)P_m(\beta)}. \qquad (2.85a)$$

5. During the addition of fractions with common denominators store/add up their numerators

$$\frac{P_{n_1}(\beta)}{P_m(\beta)} + \frac{P_{n_2}(\beta)}{P_m(\beta)} = \frac{P_{n_1}(\beta)+P_{n_2}(\beta)}{P_m(\beta)}. \qquad (2.86)$$

6. From p. 4 and 5 escape/ensues the possibility of reduction with their subsequent algebraic addition, and also the possibility of the expansion of fractions to the simplest.

7. The product of fractions exists always, whereupon

$$\left(\frac{P_{n_1}(\beta)}{P_{m_1}(\beta)}\cdot\frac{P_{n_2}(\beta)}{P_{m_2}(\beta)}\right)\varphi = P_{n_1}(\beta)\left\{\frac{1}{P_{m_1}(\beta)}\left\{P_{n_2}(\beta)\left\{\frac{1}{P_{m_2}(\beta)}\varphi\right\}\right\}\right\}. \qquad (2.87)$$

but to multiply operational fractions algebraically possible only when $n_2 \leqslant m_2$.

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If this condition is satisfied, then

$$\frac{P_{n_1}(\beta)}{P_{m_1}(\beta)}\cdot\frac{P_{n_2}(\beta)}{P_{m_2}(\beta)}=\frac{P_{n_1}(\beta)\,P_{n_2}(\beta)}{P_{m_1}(\beta)\,P_{m_2}(\beta)}=\frac{P_{n_1+n_2}(\beta)}{P_{m_1+m_2}(\beta)}. \tag{2.88}$$

Therefore, for example,

$$\frac{\beta^2}{1+\beta}\cdot\frac{1+\beta}{\beta^2}=1, \tag{2.89}$$

but

$$\frac{\beta}{1+\beta^2}\cdot\frac{1+\beta^2}{\beta}\neq 1. \tag{2.90}$$

In fact, using (1.368), it is easy to obtain

$$\frac{\beta}{1+\beta^2}\left\{\frac{1+\beta^2}{\beta}\varphi\right\}=\varphi(\eta)-\varphi(0)\cos\eta. \tag{2.91}$$

Even if $n_1 \leqslant m_1$, then the product of fractions is permutable.

8. The product of polynomial for fraction always exists, is implemented algebraically and not permutable. For example,

$$\left.\begin{aligned}&(\beta-1)\frac{\beta+1}{\beta-1}=\frac{(\beta-1)(\beta+1)}{(\beta-1)}=\beta+1\neq\frac{\beta+1}{\beta-1}(\beta-1),\\&(\beta-1)\frac{(\beta+1)^2}{\beta^2+1}=\frac{(\beta^2-1)(\beta^2+\beta+1)}{\beta^2+1}\neq\frac{(\beta+1)^2}{\beta^2+1}(\beta-1).\end{aligned}\right\} \tag{2.92}$$

For an example let us demonstrate p. 1.

Need. Let occurs (2.82), i.e., taking into account the determination of operational fraction,

$$P_{n_1}(\beta)\cdot\frac{1}{P_{m_1}(\beta)}=P_{n_2}(\beta)\cdot\frac{1}{P_{m_2}(\beta)}. \qquad (2.93)$$

Then, according to the last/latter affirmation p. 4g,

$$P_{n_1}(z)\cdot\frac{1}{P_{m_1}(z)}=P_{n_2}(z)\cdot\frac{1}{P_{m_2}(z)}, \qquad (2.94)$$

whence

$$P_{n_1}(z)P_{m_2}(z)=P_{n_2}(z)P_{m_1}(z). \qquad (2.95)$$

Products $P_{n_1}(\beta)P_{m_2}(\beta)$ and $P_{n_2}(\beta)P_{m_1}(\beta)$ satisfy conditions (2.67). Therefore from (2.95) it will follow (2.83).

Sufficiency. Let it occurs (2.83). Then, according to the last/latter affirmation p. 4 g, it is valid and (2.95), but thereby also (2.94). If we replace in (2.94) z by β, then in both parts will be obtained the formal products of the operators, which satisfy conditions (2.67). Therefore accurate it will be (2.93). Keeping in mind the determination of operational fraction, hence we obtain (2.82).

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b) Now let us consider the fractions of general view. For the operators we do not determine the action of division. Therefore the formal fraction $L_1\beta/L_2\beta$ is understood as single symbol. This operational fraction makes sense, if function L_1 (z) L_2 (z) in vicinity z = ∞ is decompose/expanded in series (2.3) or (2.4). On the basis of these series must be realized operator fraction. However, in the majority of cases above the operational fractions of general view it is possible to produce usual algebraic actions. Are justified these actions just as for rational relative to β fractions. Only, which requires certain attention, is the operation of multiplication. During its execution one should be guided (2.67). Specifically, for example:

1. If L_1 (β) L_2 (β) it is regular operator, then this fraction can be multiplied to the right by any operator and then to implement any transforms.

2. If L_1 (β) L_2 (β) it has expansion (2.53), then this fraction can be multiplied to the left by any operator and then to implement any transforms.

3. From preceding/previous it follows that

$$\frac{L_1(\beta)}{L_2(\beta)}\cdot\frac{L_3(\beta)}{L_4(\beta)}=\frac{L_1(\beta)\,L_3(\beta)}{L_2(\beta)\,L_4(\beta)} \qquad (2.96)$$

in that and only that case, when either L_1/L_2 regular operator or L_3/L_4 is represented in the form (2.53). Let us give several examples

$$\left.\begin{aligned}
&\frac{e^{t\beta}}{\cos\beta + i\sin\beta}\sqrt{\beta^2+k^2} = e^{(t-i)\beta}\sqrt{\beta^2+k^2},\\
&(\beta+\beta^2)\frac{\sin\frac{1}{\beta}}{\beta} = (1+\beta)\sin\frac{1}{\beta},\\
&\frac{\beta}{e^{-2\beta}}\cdot\frac{\cos\beta}{\beta^2} = \frac{e^{2\beta}\cos\beta}{\beta} = e^{2\beta}\cos\beta\cdot\frac{1}{\beta},\\
&\frac{\sin\beta}{\beta}\cdot\frac{\beta^2}{1-\beta} = \frac{\beta\sin\beta}{1-\beta} = \beta\cos\beta\cdot\frac{1}{1-\beta},\\
&\frac{\beta^2+k^2}{\sqrt{\beta^2+k^2}} = \sqrt{\beta^2+k^2},\qquad \frac{\beta^2}{1-\beta}\cdot\frac{\sin\beta}{\beta} \neq \frac{\beta\sin\beta}{1-\beta}.
\end{aligned}\right\}\quad (2.97)$$

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7. The algebraic actions above the mixed operators

a) If $\varphi\in\omega_z$, then series in (2.6) and (2.7) they converge absolutely. Therefore operational sum possesses the same functional properties, as the sum of the corresponding functions. For example,

$$\cos\frac{1}{\beta} + i\sin\frac{1}{\beta} = e^{\frac{i}{\beta}},$$

$$\sqrt{\beta^2+\xi^2} - \frac{\beta^2}{\sqrt{\beta^2+\xi^2}} = \frac{\xi^2}{\sqrt{\beta^2+\xi^2}}. \quad (2.98)$$

b) In the preceding/previous point/items was examined in detail the multiplication cf the operators. The basic result can be formulated thus: if operational product satisfies conditions (2.67), then it it possesses the same functional properties, as the product appropriate of functions. So that, for example,

$$\left.\begin{aligned}
&\operatorname{ch}^2\frac{1}{\beta}-\operatorname{sh}^2\frac{1}{\beta}=1,\\
&\operatorname{ch}\left(\beta+\frac{1}{\beta}\right)=\operatorname{ch}\beta\cdot\operatorname{ch}\frac{1}{\beta}+\operatorname{sh}\beta\operatorname{sh}\frac{1}{\beta}\neq\operatorname{ch}\frac{1}{\beta}\operatorname{ch}\beta+\\
&\qquad+\operatorname{sh}\frac{1}{\beta}\operatorname{sh}\beta,\\
&\frac{(\beta+1)\cos^2\beta}{\beta-1}+\frac{(\beta-1)\sin^2\beta}{\beta+1}=\frac{(\beta+1)^2\cos^2\beta}{(\beta+1)(\beta-1)}+\\
&\quad+\frac{(\beta-1)^2\sin^2\beta}{(\beta-1)(\beta+1)}=\frac{2(\beta^2+1)+2\beta\cos 2\beta}{\beta^2-1}=\\
&\qquad+\frac{1}{2}\frac{\beta^2+1}{\beta^2-1}+\frac{1}{2}\frac{\beta\cos\beta}{\beta^2-1}.
\end{aligned}\right\}\quad(2.99)$$

If product contains more than two factors, then it is expedient to use associative property (1.21). For example,

$$\beta(1+\beta)\frac{1}{\beta}(1-\beta)=\beta\cdot\left[(1+\beta)\cdot\frac{1}{\beta}\right](1-\beta)=\beta\frac{1+\beta}{\beta}(1-\beta)=$$
$$=\frac{\beta(1+\beta)}{\beta}(1-\beta)=(1+\beta)(1-\beta)=1-\beta^2. \quad (2.100)$$

c) from p. a and b follows that above the mixed operators are permissible any algebraic actions, connected with addition and multiplication, if only with multiplications are implemented condition (2.67).

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It is natural that in appendices of the action above the operators they are implemented on purely formal basis. If we in this case forget about the need for satisfaction of conditions (2.67), then it is possible to allow large error. For example, although at first glance it seems that

$$\left(\frac{1}{\beta}\cos\beta\right)^2 = \frac{\cos^2\beta}{\beta^2}, \qquad \left(\beta\,\mathrm{ch}\,\frac{1}{\beta}\right)^2 = \beta^2\,\mathrm{ch}^2\,\frac{1}{\beta}, \tag{2.101}$$

but in actuality

$$\left.\begin{aligned} &\left(\frac{1}{\beta}\cos\beta\right)^2\varphi = \frac{1}{\beta}\cos\beta\left\{\frac{1}{\beta}\cos\varphi\right\} = \\ &= \frac{\cos^2\beta}{\beta^2}\varphi + \sum_{s=0}^{\infty}\frac{(-1)^s}{[2(s+1)]!}\,!\;\varphi^{2s}(0), \\ &\left(\beta\,\mathrm{ch}\,\frac{1}{\beta}\right)^2\varphi = \beta\,\mathrm{ch}\,\frac{1}{\beta}\left\{\beta\,\mathrm{ch}\,\frac{1}{\beta}\,\varphi\right\} = \\ &= \beta^2\,\mathrm{ch}^2\,\frac{1}{\beta}\,\varphi - \frac{\varphi(0)}{2}\sum_{s=0}^{\infty}\frac{\eta^{2s}}{(s+1)\,[(2s)!]^2}. \end{aligned}\right| \tag{2.102}$$

To this result it is easy to arrive, using formula (2.64).

Can arise inaccuracies, also, during the carrying out of factors for brackets. For example, while the actions

$$\left.\begin{array}{l}
\beta\cos\beta+\beta^{2}\sin\beta=\beta(\cos\beta+\beta\sin\beta)=(\cos\beta+\beta\sin\beta)\beta,\\
\beta\cos\beta+\beta^{2}\sin\frac{1}{\beta}=\beta\left(\cos\beta+\beta\sin\frac{1}{\beta}\right),\\
\frac{1+\beta^{2}}{1-\beta^{2}}+\frac{e^{\beta}}{1-\beta}=\left(\frac{1+\beta^{2}}{1+\beta}+e^{\beta}\right)\cdot\frac{1}{1-\beta},\\
\frac{\sin^{2}\frac{1}{\beta}}{1-\beta}+\sin\frac{1}{\beta}=\left(\sin\frac{1}{\beta}+1-\beta\right)\frac{\sin\frac{1}{\beta}}{1-\beta},\\
\frac{\sin^{2}\frac{1}{\beta}}{1-\beta}+\sin\frac{1}{\beta}=\left(\frac{\sin\frac{1}{\beta}}{1-\beta}+1\right)=\sin\frac{1}{\beta}\left(\frac{\sin\frac{1}{\beta}}{1-\beta}+1\right)
\end{array}\right\} \quad (2.103)$$

are completely valid, the following transforms are inaccurate:

$$\left.\begin{array}{l}
\beta\cos\beta+\beta^{2}\sin\frac{1}{\beta}=\left(\cos\beta+\beta\sin\frac{1}{\beta}\right)\beta,\\
\frac{\beta+1}{\beta^{2}+1}+(1+\beta)e^{\beta}=\left(\frac{1}{\beta^{2}+1}+e^{\beta}\right)(\beta+1),\\
\frac{\sin\beta}{1-\beta}+\frac{\cos\beta}{1-\beta}=\frac{1}{1-\beta}(\sin\beta-\cos\beta).
\end{array}\right\} \quad (2.104)$$

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8. Examples of the realization of the mixed operators [1].

FOOTNOTE [1]. The significant part of the given here results was obtained together with N. A. Wenzel. ENDFOOTNOTE.

a) let

$$L(\beta)=\sqrt{\beta^2-k^2}. \qquad (2.104a)$$

Function L (z) $=\sqrt{z^2-k^2}$ has a branch point in z = ±k, but is one-sheeted in z = ∞, which is for it simple pole. Therefore

$$L(\beta)=\beta\left(1-\frac{k^2}{\beta^2}\right)^{\frac{1}{2}}=\beta-\sum_{s=1}^{\infty}\frac{(2s-3)!!}{2^s\cdot s!}\cdot\frac{k^{2s}}{\beta^{2s-1}}, \qquad (2.105)$$

or, since

$$(2s-3)!! = \frac{(2s-3)!!\,(2s-2)!!}{(2s-2)!!} = \frac{(2s-2)!}{2^{s-1}\cdot(s-1)!}, \quad (2.106)$$

that

$$L(\beta) = \beta - \sum_{s=1}^{\infty} \frac{(2s-2)!}{2^{2s-1}\cdot(s-1)!\,s!}\cdot\frac{k^{2s}}{\beta^{2s-1}}. \quad (2.107)$$

Consequently,

$$a_{-2s} = 0, \quad a_{-(2s-1)} = -\frac{(2s-2)!\,k^{2s}}{2^{2s-1}\cdot(s-1)!\,s!}, \quad s = 1, 2, 3, \ldots \quad (2.108)$$

Therefore

$$\sum_{s=1}^{\infty} \frac{a_{-s}(\eta-\zeta)^{s-1}}{(s-1)!} = -\sum_{s=1}^{\infty} \frac{(2s-2)!\,k^{2s}\cdot(\eta-\zeta)^{2s-2}}{2^{2s-1}\cdot(s-1)!\,s!\,(2s-2)!} = -\frac{k}{\eta-\zeta}\sum_{s=0}^{\infty}\frac{[k(\eta-\zeta)]^{2s+1}}{2^{2s+1}\cdot s!\,(s+1)!}. \quad (2.109)$$

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Using formula (6.457) from [37], we can write

$$\sum_{s=1}^{\infty} \frac{a_{-s}(\eta-\zeta)^{s-1}}{(s-1)!} = \frac{k}{\zeta-\eta} J_1[k(\eta-\zeta)], \quad (2.110)$$

where J_1 - the Bessel function of first-order of apparent/imaginary argument. Then from (2.6) we obtain

$$\boxed{\sqrt{\beta^2-k^2}\,\varphi = \varphi'(\eta) - k\int_0^{\eta}\frac{\varphi(\zeta)}{\eta-\zeta}J_1[k(\eta-\zeta)]\,d\zeta,} \quad (2.111)$$

whence, after replacing k by ik,

$$\sqrt{\beta^2+k^2}\varphi=\varphi'(\eta)-k\int_0^\eta\frac{\varphi(\zeta)}{\eta-\zeta}J_1[k(\eta-\zeta)]\,d\zeta. \quad (2.112)$$

b) Let

$$L(\beta)=e^{k/\beta}. \quad (2.113)$$

Function L (z) = exp (k/z) has essential singularity into z = 0 and is regular in the remaining part of the plane. Therefore

$$L(\beta)=1+\sum_{s=1}^{\infty}\frac{k^s}{s!}\cdot\frac{1}{\beta^s} \quad (2.114)$$

and

$$\sum_{s=1}^{\infty}\frac{a_{-s}(\eta-\zeta)^{s-1}}{(s-1)!}=\sum_{s=1}^{\infty}\frac{k^s(\eta-\zeta)^{s-1}}{(s-1)!\,s!}=k\sum_{n=0}^{\infty}\frac{\left(\frac{2\sqrt{k(\eta-\zeta)}}{2}\right)^{2n}}{n!\,(n+1)!}= \quad (2.115)$$

$$=\frac{k}{\sqrt{k(\eta-\zeta)}}\sum_{s=1}^{\infty}\frac{\left(\frac{2\sqrt{k(\eta-\zeta)}}{2}\right)^{2s+1}}{s!\,(s+1)!}=\frac{k}{\sqrt{k(\eta-\zeta)}}J_1[2\sqrt{k(\eta-\zeta)}].$$

Here we again used formula (6.457) from [37]. By introducing (2.115)

in (2.6) and by taking into account (2.114), we will obtain

$$e^{\frac{k}{\beta}}\varphi = \varphi(\eta) + k\int_0^\eta \frac{I_1[2\sqrt{k(\eta-\zeta)}]}{\sqrt{k(\eta-\zeta)}}\varphi(\zeta)\,d\zeta. \qquad (2.116)$$

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By utilizing the formula

$$\frac{dI_0[2\sqrt{k(\eta-\zeta)}]}{d\zeta} = -k\frac{I_1[2\sqrt{k(\eta-\zeta)}]}{\sqrt{k(\eta-\zeta)}} \qquad (2.117)$$

and after fulfilling then integration in parts, it is possible formula (2.116) to convert to the form

$$\boxed{e^{\frac{k}{\beta}}\varphi = \varphi(0)\,I_0(2\sqrt{k\eta}) + \int_0^\eta I_0(2\sqrt{k(\eta-\zeta)}\,\varphi'(\zeta)\,d\zeta.} \qquad (2.118)$$

By substituting in (2.116) k on - k and by taking the linear combination of results, let us find

$$\left.\begin{aligned} \mathrm{ch}\frac{k}{\beta}\varphi &= \varphi(\eta) + \\ &+ k\int_0^\eta \frac{J_1(2\sqrt{k(\eta-\zeta)}) - J_1(2\sqrt{k(\eta-\zeta)}}{2\sqrt{k(\eta-\zeta)}}\varphi(\zeta)\,d\zeta. \\ \mathrm{sh}\frac{k}{\beta}\varphi &= \\ &= k\int_0^\eta \frac{J_1(2\sqrt{k(\eta-\zeta)}) + J_1(2\sqrt{k(\eta-\zeta)})}{2\sqrt{k(\eta-\zeta)}}\varphi(\zeta)\,d\zeta. \end{aligned}\right\} \qquad (2.119)$$

c) Let now

$$L(\beta) = e^{\xi(\beta - \sqrt{\beta^2+k^2})}. \tag{2.120}$$

Function L (z) = exp [ξ (z - $\sqrt{z^2-k^2}$)] has branch points in z = ±ki and is regular in the remaining part of plane, whereupon L (∞) = 1. Therefore its expansion in the vicinity of the infinitely receded point will take the form:

$$L(z) = 1 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \ldots. \tag{2.121}$$

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In order to obtain this expansion, let us write first

$$\left.\begin{aligned} &\sqrt{z^2+k^2} = z\left(1+\frac{k^2}{z^2}\right)^{\frac{1}{2}} = z - \sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{(2\nu)!\,k^{2(\nu+1)}}{2^{2\nu+1}\nu!\,(\nu+1)!}\times \\ &\qquad \times\frac{1}{z^{2\nu+1}}, \\ &(z-\sqrt{z^2+k^2})^n = (-1)^n n\,(kz)^n \times \\ &\times\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{(n+2\nu-1)!}{\nu!\,(n+\nu)!}\cdot\left(\frac{k}{2z}\right)^{n+2\nu}, \quad n=1,2,\ldots \end{aligned}\right\} \tag{2.122}$$

Then

$$\begin{aligned} L(\beta) &= 1 + \sum_{n=0}^{\infty}\frac{\xi^{n+1}}{(n+1)!}(\beta-\sqrt{\beta^2+k^2})^{n+1} = \\ &= 1 - k\xi\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{k\xi}{2}\right)^n\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{(n+2\nu)!}{(n+1+\nu)!\,\nu!}\times \\ &\qquad \times\frac{k^{n+2\nu+1}}{2^{2\nu+1}}\cdot\frac{1}{\beta^{n+2\nu+1}}. \end{aligned} \tag{2.123}$$

Put

$$\frac{1}{\beta^{n+2\nu+1}}\varphi(\eta)=\frac{1}{(n+2\nu)!}\int_0^\eta(\eta-\zeta)^{n+2\nu}\varphi(\zeta)\,d\zeta. \qquad (2.124)$$

Therefore

$$L(\beta)\varphi=\varphi(\eta)-k\xi\sum_{n=0}^{\infty}(-1)^n\frac{k^n\xi^n}{n!\,2^n}\sum_{\nu=0}^{\infty}(-1)^\nu\frac{k^{n+2\nu+1}}{(n+1+\nu)!\,\nu!}\times$$
$$\times\frac{1}{2^{2\nu+1}}\int_0^\eta(\eta-\zeta)^{n+2\nu}\varphi(\zeta)\,d\zeta. \qquad (2.125)$$

If $\varphi(\eta)$ is integrated, then the entering here series converge absolutely and evenly. By using this, it is possible to lead the transformations of the series, as a result of which let us arrive at

$$e^{\xi(\beta-\sqrt{\beta^2+k^2})}\varphi(\eta)=\varphi(\eta)-$$
$$-k\xi\int_0^\eta\left\{\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\,(s+1)!}\left[\frac{k}{2}\sqrt{(\eta-\zeta)(\eta-\zeta+2\xi)}\right]^{2s+1}\right\}\times \qquad (2.126)$$
$$\times\frac{\varphi(\zeta)\,d\zeta}{\sqrt{(\eta-\zeta)(\eta-\zeta+2\xi)}}.$$

or, according to (6.451) from [37],

$$e^{\xi(\beta-\sqrt{\beta^2+k^2})}\varphi=\varphi(\eta)-k\xi\int_0^\eta\frac{J_1\left[\frac{k}{2}\sqrt{(\eta-\zeta)(\eta-\zeta+2\xi)}\right]}{\sqrt{(\eta-\zeta)(\eta-\zeta+2\xi)}}\varphi(\zeta)\,d\zeta. \qquad (2.127)$$

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Bearing in mind that

$$e^{-\xi\beta}\cdot e^{\xi(\beta-\sqrt{\beta^2+k^2})}=e^{-\xi\sqrt{\beta^2+k^2}}, \qquad (2.128)$$

and acting on both parts (2.127) by operator $e^{-\xi D}$, we obtain

$$\boxed{e^{-\xi\sqrt{\beta^2+k^2}}\varphi=\varphi(\eta-\xi)+k\xi\int_{\eta-\xi}^{\eta}\frac{J_1\left[\frac{k}{2}\sqrt{(\eta-\zeta)^2-\xi^2}\right]}{\sqrt{(\eta-\zeta)^2-\xi^2}}\varphi(\zeta)\,d\zeta.} \quad (2.129)$$

Hence it is possible to find the realization of the different transcendental operators. For example, by substituting ξ by $i\xi$, then on $-i\xi$ and by store/adding up results, let us have

$$\boxed{\begin{aligned}\cos\xi\sqrt{\beta^2+k^2}\varphi&=\frac{\varphi(\eta+i\xi)+\varphi(\eta-i\xi)}{2}+\\&+ik\xi\int_{\eta-i\xi}^{\eta+i\xi}\frac{J_1\left[\frac{k}{2}\sqrt{(\eta-\zeta)^2+\xi^2}\right]}{\sqrt{(\eta-\zeta)^2+\xi^2}}\varphi(\zeta)\,d\zeta.\end{aligned}} \quad (2.130)$$

§10. Separate observations about the operators.

1. Expansion of the domain of definition of operator.

In §5, 7 and 9 were establish/installed the domains of definition for the regular, singular and mixed operators. These

domains can be considerably expanded, if we use the method, used in §4. By comparing §4 with §3, 5 and by all following, it is possible to plan two possible approaches to the construction of the theory of the operators: a) is assigned concrete/specific/actual operation, is located analytical expression for an operator, who realize/accomplishes this operation, and are investigated his properties; b) is assigned analytical expression for an operator, are investigated its properties and is found the corresponding to it operation (operator's realization).

The first method is only casually touched upon in §4. The advantage of this method of the introduction of the operators is the considerable expansion of the domain of definition of operator to its "natural" boundaries, which depend only on operation, but not on operator. But if we now use the formulas for the realization of the operators, found in §6, 8 and 9, and to take these formulas for the determination of the corresponding operators, then for the domain of determination of these operators it is possible to take many functions, for which is permissible this operation. Thus, for instance, according to (1.246), by domain of definition for operator $\sin\beta/\beta$ will be not $\omega^{(A)}$, but many functions, integrated along segment $[\eta - l; \eta + l]$.

2. The equations, by which satisfy the values of operator-functions.

In p. 8 §6, it was shown, that the values of operator-functions satisfy some differential equations. This fact can be interpreted even by another form. Let us consider that in equality $L(\xi, \beta)\,\phi(\eta) = \Phi(\xi,\eta)$, known to L and Φ. Then this equality is be certain equation (transcendental differential, integrodifferential infinite order, differential-difference and so forth) relative to unknown function ϕ. The formulas, similar (1.254) and (1.256), show that the right side of this equation cannot be represented by arbitrary function.

The aforesaid is related not only to regular, but also generally to the mixed operators. For confirmation let us consider the following of two examples.

a) let

$$e^{\frac{\xi}{\beta}}\varphi(\eta)=\Phi(\xi,\eta). \tag{2.131}$$

We differentiate with respect to ξ

$$\frac{1}{\beta}e^{\frac{\xi}{\beta}}\varphi=\frac{\partial\Phi}{\partial\xi}. \tag{2.132}$$

Differentiating the latter from η and taking into account the

validity of the equality

$$\beta \cdot \frac{1}{\beta} e^{\frac{\xi}{\beta}} = e^{\frac{\xi}{\beta}}, \qquad (2.133)$$

let us find

$$e^{\frac{\xi}{\beta}} \varphi = \frac{\partial^2 \Phi}{\partial \xi \partial \eta}. \qquad (2.134)$$

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From (2.131) and (2.134) now we obtain

$$\frac{\partial^2 \Phi}{\partial \xi \partial \eta} - \Phi = 0. \qquad (2.135)$$

b) Let

$$\sqrt{\beta^2 + \xi^2}\, \varphi(\eta) = \Phi(\xi, \eta). \qquad (2.136)$$

We differentiate with respect to ξ

$$\frac{\xi}{\sqrt{\beta^2 + \xi^2}} \varphi = \frac{\partial \Phi}{\partial \xi}. \qquad (2.137)$$

Then

$$\frac{\beta^2}{\sqrt{\beta^2 + \xi^2}} \varphi = \left(\sqrt{\beta^2 + \xi^2} - \frac{\xi^2}{\sqrt{\beta^2 + \xi^2}}\right) \varphi = \Phi - \xi \frac{\partial \Phi}{\partial \xi}. \qquad (2.138)$$

Twice by differentiating (2.137) with respect to η, by multiplying (2.138) by $-\xi$ and by store/adding up results, let us find

$$\frac{\partial^3 \Phi}{\partial \xi \partial \eta^2} + \xi^2 \frac{\partial \Phi}{\partial \xi} - \xi \Phi = 0. \qquad (2.139)$$

3. Some identities for Bessel functions.

The formulas of the realization of the singular and mixed operators make it possible to establish/install some new identities for Bessel functions. Let us consider two examples, based on the application/use of formulas (1.390) and (2.107).

The expansion

$$\frac{1}{\sqrt{\beta^2+k^2}}=\frac{1}{\beta}-\frac{1}{2}\frac{k^2}{\beta^3}+\ldots,\ \sqrt{\beta^2+k^2}=\beta+\frac{1}{2}\frac{k^2}{\beta}-\ldots \tag{2.140}$$

shows that operator $\frac{1}{\sqrt{\beta^2+k^2}}$ - singular, and $\sqrt{\beta^2+k^2}$ - mixed. That means these operators do not commutate, but in accordance with (2.67)

$$\sqrt{\beta^2+k^2}\cdot\frac{1}{\sqrt{\beta^2+k^2}}=1;\ \frac{1}{\sqrt{\beta^2+k^2}}\cdot\sqrt{\beta^2+k^2}\neq 1. \tag{2.141}$$

Therefore

$$\sqrt{\beta^2+k^2}\left\{\int_0^\eta \varphi(\zeta)J_0[k(\eta-\zeta)]\,d\zeta\right\}=\varphi(\eta).$$

i. e.,

$$\frac{d}{d\eta}\int_0^{\eta}\varphi(\zeta)J[k(\eta-\zeta)]\,d\zeta+k\int_0^{\eta}\frac{J_1[k(\eta-\zeta)]}{\eta-\zeta}\,d\zeta\int_0^{\zeta}\varphi(t)J_0[k(\zeta-t)]\,dt=\varphi(\eta). \quad (2.142)$$

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After fulfilling differentiation and by varying the order of integration, let us arrive at the identity

$$\int_0^{\eta}\varphi(t)\,dt\int_t^{\eta}\frac{J_0[k(\zeta-t)]\,J_1[k(\eta-\zeta)]}{\eta-\zeta}\,d\zeta=\int_0^{\eta}\varphi(\zeta)J_1[k(\eta-\zeta)]\,d\zeta. \quad (2.143)$$

where ϕ - arbitrary function.

Further, taking into account (1.390), (1.368) and that that

$$\frac{1}{\sqrt{\beta^2+k^2}}\cdot\frac{1}{\sqrt{\beta^2+k^2}}=\frac{1}{\beta^2+k^2}, \quad (2.144)$$

we will obtain

$$\int_0^{\eta}\varphi(t)\,dt\int_t^{\eta}J_0[k(\eta-\zeta)]\,J_0[k(\zeta-t)]\,d\zeta=\frac{1}{k}\int_0^{\eta}\sin k(\eta-\zeta)\,\varphi(\zeta)\,d\zeta. \quad (2.145)$$

whence, in particular,

$$\int_0^{\eta} dt \int_t^{\eta} J_0 k[(\eta-\zeta)] J_0[k(\zeta-t)]\, d\zeta = \frac{1-\cos k\eta}{k^2}. \quad (2.146)$$

4. On the operators L (α, β).

In the same way as this was done for the operators L (β), the operators L (α, β), where

$$\alpha = \frac{d}{d\xi},\ \beta = \frac{d}{d\eta}, \quad (2.147)$$

or

$$\alpha = \frac{\partial}{\partial\xi},\ \beta = \frac{\partial}{\partial\eta}, \quad (2.148)$$

it is possible to determine by the formula

$$L(\alpha,\beta) \longleftrightarrow L(z, w); \quad (2.149)$$

here L (z, w) is analytic function of two complex variables.

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But in the method of the initial functions for two-dimensional tasks with such operators barely it is necessary to deal (we will be met them only in §16). Therefore, in no way by affecting the theory of the operators L (α, β), we will be bounded only to several formulas for the realization of such operators. These formulas will be obtained from already being by purely formal way. Of the authenticity of results it is possible to be convinced by direct checking.

For polynomial operators the operator's value is located directly, for example,

$$\Phi(\xi, \eta) = (\alpha^4 - 2\alpha^2\beta^2 + \beta^4)\varphi(\xi, \eta) = \frac{\partial^4\varphi}{\partial\xi^4} - 2\frac{\partial^4\varphi}{\partial\alpha^2\partial\beta^2} + \frac{\partial^4\varphi}{\partial\eta^4}. \tag{2.150}$$

For exponential operator, according to (1.236), we will obtain

$$e^{m\alpha+n\beta}\varphi(\xi, \eta) = \varphi(\xi + m, \eta + n) = \Phi(\xi, \eta). \tag{2.151}$$

Now let us consider operator $1/(\alpha + \beta)$. From formula (1.352) with n = 1, let us have

$$\frac{1}{\beta + k}\varphi(\eta) = \int_0^\eta e^{k(\zeta-\eta)}\varphi(\zeta)\,d\zeta, \tag{2.152}$$

or

$$\frac{1}{\alpha + k}\varphi(\xi) = \int_0^\xi e^{k(\zeta-\xi)}\varphi(\zeta)\,d\zeta. \tag{2.153}$$

After replacing in (2.152) k on α either in (2.153) k on β and after taking into account (1.251), let us find

$$\Phi(\xi, \eta)=\frac{1}{\alpha+\beta}\varphi(\xi, \eta)=\int_0^{\eta} e^{(\zeta-\eta)\alpha}\varphi(\xi, \zeta)\,d\zeta=\int_0^{\eta}\varphi(\xi+\zeta-\eta, \zeta)\,d\zeta. \tag{2.154}$$

or

$$\Phi(\xi, \eta)=\frac{1}{\alpha+\beta}\varphi(\xi, \eta)=\int_0^{\xi} e^{(\zeta-\xi)\beta}\varphi(\zeta, \eta)\,d\zeta=\int_0^{\xi}\varphi(\zeta, \eta+\zeta-\xi)\,d\zeta. \tag{2.155}$$

Is checked formula (2.155). We have:

$$\frac{\partial\Phi}{\partial\xi}=\varphi(\xi, \eta)-\int_0^{\xi}\left.\frac{\partial\varphi(u, v)}{\partial v}\right|_{\substack{u=\zeta\\ v=\eta+\zeta-\xi}} d\zeta, \quad \frac{\partial\Phi}{\partial\eta}=\int_0^{\xi}\left.\frac{\partial\varphi(u, v)}{\partial v}\right|_{\substack{u=\zeta\\ v=\eta+\zeta-\xi}} d\zeta. \tag{2.156}$$

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Hence it is apparent that as probably

$$(\alpha+\beta)\left\{\frac{1}{\alpha+\beta}\varphi\right\}=(\alpha+\beta)\Phi(\xi, \eta)=\frac{\partial\Phi}{\partial\xi}+\frac{\partial\Phi}{\partial\eta}=\varphi(\xi, \eta). \tag{2.157}$$

In this example it is evident that unlike $L^-(\beta)$ the singular operators $L^-(\alpha, \beta)$ have on several forms of realization. So, two representations can be obtained also for

$$\Phi(\xi, \eta)=\frac{1}{\alpha^2+\beta^2}\varphi(\xi, \eta). \tag{2.158}$$

According to (1.368)

$$\frac{1}{\beta^2+k^2}\varphi(\eta)=\int_0^{\eta}\frac{\sin k(\eta-\zeta)}{k}\varphi(\zeta)\,d\zeta \tag{2.159}$$

or

$$\frac{1}{\alpha^2+k^2}\varphi(\xi)=\int_0^{\xi}\frac{\sin k(\xi-t)}{k}\varphi(t)\,dt. \tag{2.160}$$

After replacing in (2.159) k on α either in (2.160) k on β and taking into account (1.246) let us find

$$\Phi(\xi, \eta) = \frac{1}{\alpha^2+\beta^2}\varphi(\xi, \eta) = \int_0^{\eta} \frac{\sin(\eta-\zeta)\alpha}{\alpha}\varphi(\xi, \zeta)\,d\zeta = \\ = \frac{1}{2i}\int_0^{\eta} d\zeta \int_{\xi-i(\eta-\zeta)}^{\xi+i(\eta-\zeta)} \varphi(t, \zeta)\,dt. \tag{2.161}$$

or

$$\Phi(\xi, \eta) = \frac{1}{\alpha^2+\beta^2}\varphi(\xi, \eta) = \int_0^{\xi} \frac{\sin(\xi-t)\beta}{\beta}\varphi(t, \eta)\,dt = \\ = \frac{1}{2i}\int_0^{\xi} dt \int_{\eta-i(\xi-t)}^{\eta+i(\xi-t)} \varphi(t, \zeta)\,d\zeta. \tag{2.162}$$

It is not difficult to ascertain that for both expressions it will be

$$(\alpha^2+\beta^2)\left\{\frac{1}{\alpha^2+\beta^2}\varphi(\xi, \eta)\right\} = (\alpha^2+\beta^2)\Phi = \frac{\partial^2\Phi}{\partial\xi^2} + \frac{\partial^2\Phi}{\partial\eta^2} = \varphi(\xi, \eta). \tag{2.163}$$

Chapter 3.

METHOD OF THE INITIAL FUNCTIONS FOR TWO-DIMENSIONAL BOUNDARY-VALUE PROBLEMS.

§11. Cauchy Problem.

The present chapter is dedicated to boundary-value problems, but for a larger clarity first will be in detail examined Cauchy's elementary problem (about vibration of infinite string).

Let the function $u(\xi,\eta)$ in half-plane $0 < \xi < \infty, -\infty < \eta + \infty$, satisfy the equation

$$\frac{\partial^2 u}{\partial \xi^2} = a^2 \frac{\partial^2 u}{\partial \eta^2}, \quad a = \text{const.} \qquad (3.1)$$

Let us name boundary $\xi = 0$ initial line (Fig. 1). On it they can be assigned/prescribed: function and its normal derivative, i. e.,

$$U_0 = U_0(\eta) = u(0, \eta), \; U_1 = U_1(\eta) = \left(\frac{\partial u}{\partial \xi}\right)_{\xi=0}. \qquad (3.2)$$

Let us call these values the initial functions. The value of function u at each point of domain depends, on one hand, on the coordinates of

this point, and with another, on the initial functions. Dependence of u on ξ and η is expressed by usual functional dependence. Dependence u on U_0 and U_1 is more complex: it is possible to count what $u(\xi,\eta)$ is the result of the action of some operators above functions U_0 and U_1. Accordingly, let us seek solution in the form

$$u(\xi, \eta) = L_0(\xi, \beta) U_0(\eta) + L_1(\xi, \beta) U_1(\eta), \tag{3.3}$$

L_0 and L_1 - to be determined of operator-function.

Let us substitute (3.3) in (3.1), after explaining preliminarily the method of differentiation of the values of operator-functions. According to (1.149) or (1.315)

$$\frac{\partial^2}{\partial \xi^2}\Big[L(\xi, \beta) U(\eta)\Big] = \frac{d^2 L}{d\xi^2}\Big\{U(\eta)\Big\}, \tag{3.4}$$

and on the strength of the determination of the operator

$$\frac{\partial^2}{\partial \eta^2}\Big[L(\xi, \beta) U(\eta)\Big] = \beta^2 L(\xi, \beta) U(\eta) = (\beta^2 L)\{U(\eta)\}. \tag{3.5}$$

Therefore from (3.1) we will obtain

$$\frac{d^2 L_0}{d\xi^2} U_0 + \frac{d^2 L_1}{\partial \xi^2} U_1 = a^2 (\beta^2 L_0 U_0 + \beta^2 L_1 U_1) \tag{3.6}$$

or

$$\left(\frac{d^2 L_0}{d\xi^2} - a^2\beta^2 L_0\right) U_0 + \left(\frac{d^2 L_1}{d\xi^2} - a^2\beta^2 L_1\right) U_1 = 0. \tag{3.7}$$

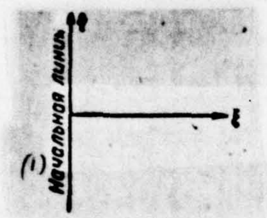

Fig. 1.

Key: (1). Initial lines.

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Functions U_0 and U_1 are independent (for example, retaining U_0, it is possible arbitrarily to change U_1), and this means that

$$\left(\frac{d^2L_j}{d\xi^2}-a^2\beta^2L_j\right)U_j=0, \quad j=0, 1. \tag{3.8}$$

Here U_0 and U_1-are arbitrary functions. Hence in accordance with the determination of zero operator (page 28) let us have

$$\frac{d^2L_j}{d\xi^2}-a^2\beta^2L_j=0. \tag{3.9}$$

Integrating these equations (see Section d, page 53), we find

$$L_0(\xi, \beta)=A_0\operatorname{ch}a\xi\beta+B_0\operatorname{sh}a\xi\beta, \quad L_1(\xi, \beta)=A_1\operatorname{ch}a\xi\beta+B_1\operatorname{sh}a\xi\beta. \tag{3.10}$$

For determining constants (operators) A_l and B_l let us introduce (3.3) in (3.2):

$$u_0(\eta) = L_0(0,\ \beta)\,U_0(\eta) + L_1(0,\ \beta)\,U_1(\eta),$$
$$U_1(\eta) = \frac{dL_0}{\partial\xi}\bigg|_{\xi=0} U_0(\eta) + \frac{dL_1}{d\xi}\bigg|_{\xi=0} U_1(\eta). \qquad (3.11)$$

whence (again taking into account independence U_0 and U_1)

$$L_0(0,\ \beta) = 1,\quad L_1(0,\ \beta) = 0, \qquad (3.12)$$
$$\frac{dL_0}{d\xi}\bigg|_{\xi=0} = 0,\quad \frac{dL_1}{d\xi}\bigg|_{\xi=0} = 1.$$

But hyperbolic operator-functions possess the properties of usual hyperbolic functions; therefore from (3.10) and (3.12) we will obtain:

$$A_0 = 1\quad A_1 = 0,$$
$$B_0 a\beta = 0,\quad B_1 a\beta = 1. \qquad (3.13)$$

Hence, after multiplying the last/latter equality to the right on $1/\beta$, let us find

$$A_0 = 1,\ B_0 = 0,\ A_1 = 0,\ B_1 = \frac{1}{a\beta}. \qquad (3.14)$$

and, therefore,

$$L_0(\xi,\ \beta) = \operatorname{ch} a\xi\beta,\ L_1(\xi,\ \beta) = \frac{\operatorname{sh} a\xi\beta}{a\beta}. \qquad (3.15)$$

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After substituting these expressions in (3.3), let us have

$$u(\xi, \eta) = \operatorname{ch} a\xi\beta U_0(\eta) + \frac{\operatorname{sh} a\xi\beta}{a\beta} U_1(\eta). \tag{3.16}$$

If the initial functions are assigned/prescribed:

$$U_0 = \varphi(\eta), \quad U_1 = \psi(\eta) \tag{3.17}$$

(are known the deviations of string and speed of its points to zero time), then of (3.16) we will obtain the solution of problem in the operational form

$$u(\xi, \eta) = \operatorname{ch} a\xi\beta\varphi(\eta) + \frac{\operatorname{sh} a\xi\beta}{a\beta}\psi(\eta). \tag{3.18}$$

In order to pass of operational form to usual, it is necessary to use the obtained in the first chapters formulas for the realization of the operators. So, after using (1.237) and (1.245) to (3.18), let us arrive at d' Alembert's known formula

$$u(\xi, \eta) = \frac{1}{2}\left[\varphi(\eta + a\xi) + \varphi(\eta - a\xi)\right] + \frac{1}{2a}\int_{\eta - a\xi}^{\eta + a\xi} \psi(\zeta)\, d\zeta. \tag{3.19}$$

In this task operators L_0 and L_1 were obtained regularly, so that, strictly speaking (p. 2 §5), the initial functions must belong to class $\omega^{(A)}$. However, by taking into account the aforesaid in p. 1 §10 and §4, it is possible to claim that during function U_0 and U_1 is superimposed the only limitation: they must allow/assume the

operations, determined by formula (3.19).

§12. General solution of two-dimensional boundary-value problem.

Let the unknown function u (ξ, η) in rectangular domain $0 \leqslant \xi \leqslant 1$, $0 \leqslant \eta \leqslant \lambda$ (Fig. 2) satisfy equation in particular derived

$$D(\alpha, \beta)\{u(\xi, \eta)\} = f(\xi, \eta). \tag{3.20}$$

where f - assigned function, D - the assigned differential operator of order n. Let us consider that this is operator with constant coefficients, i. e.,

$$D(\alpha, \beta) = \sum_{k+s=0}^{k+s=n} a_{ks}\alpha^k\beta^s. \tag{3.21}$$

Let us take boundary $\xi = 0$ for the initial line. On it can be assigned/prescribed the functions

$$U_j(\eta) = \left(\frac{\partial^j u}{\partial \xi^j}\right)_{\xi=0}, \quad j = 0, 1, \ldots, n-1, \tag{3.22}$$

or the making determined physical sense linear combinations of these functions

$$U_j(\eta) = \sum_{s=0}^{n-1} b_{js}\left(\frac{\partial^s u}{\partial \xi^s}\right)_{\xi=0}, \quad j = 0, 1, \ldots, n-1 \tag{3.23}$$

(for example, with the curvature of plate such linear combinations

they will be the boundary/edge bending moments and the given transverse forces).

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In both cases let us name $U_l(\eta)$ the initial functions and let us consider that the unknown function $u(\xi, \eta)$ is the "function" of these initial functions. Accordingly the general solution let us seek in the form

$$u(\xi, \eta) = \sum_{l=0}^{n-1} L_l(\xi, \beta) U_l(\eta) + u_f. \qquad (3.24)$$

Here $L_l(\xi, \beta)$ — to be determined of operator-function, a u_f — the particular solution to equation (3.20), whereupon such, which turns on the initial line into zero together with its derivatives to (n - 1) th order inclusively, i.e.,

$$\left.\frac{\partial^s u_f}{\partial \xi^s}\right|_{\xi=0} = 0, \quad s = 0, 1, \ldots, n-1. \qquad (3.25)$$

Solution (3.24), is named common/general/total in the sense that it is not connected with specific boundary conditions.

Now it is necessary to substitute expression (3.24) into equation (3.20). Preliminarily let us note that, according to (1.141) and (1.315),

$$\left.\begin{array}{c} \alpha^k \{L(\xi, \beta) U(\eta)\} = \dfrac{\partial^k}{\partial \xi^k}\{L(\xi, \beta) U(\eta)\} = \dfrac{d^k L(\xi, \beta)}{d\xi^k}\{U(\eta)\}, \\ \beta^s \{L(\xi, \beta) U(\eta)\} = \beta^s L(\xi, \beta)\{U(\eta)\}, \\ \alpha^k \beta^s \{L(\xi, \beta) U(\eta)\} = \beta^s \dfrac{d^k L(\xi, \beta)}{d\xi^k}\{U(\eta)\}. \end{array}\right\} (3.26)$$

~~A~~ and, which means,

$$D(\alpha,\ \beta)\{L(\xi,\ \beta)U(\eta)\} = \sum_{h+s=0}^{h+s=n} a_{hs}\beta^s \frac{d^h L(\xi,\ \beta)}{d\xi^h}\{U(\eta)\} = \tag{3.27}$$

$$= \left[\left(\sum_{h+s=0}^{h+s=n} a_{hs}\beta^s \frac{d^h}{d\xi^h}\right) L(\xi,\ \beta)\right]\{U(\eta)\} =$$

$$= \left[D\left(\frac{d}{d\xi},\ \beta\right) L(\xi,\ \beta)\right]\{U(\eta)\}.$$

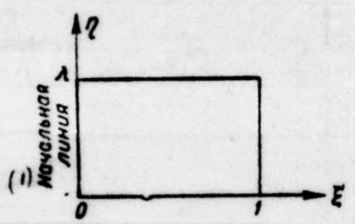

Fig. 2.

Key: (1). Initial line.

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Therefore, after substituting (3.24) in (3.20), let us have

$$\sum_{j=0}^{n-1}\left[D\left(\frac{d}{d\xi},\ \beta\right) L_j(\xi,\ \beta)\right]\{U_j(\eta)\} = 0. \tag{3.28}$$

But the initial functions are independent (each of them can be assigned/prescribed independent of others), therefore,

$$\left[D\left(\frac{d}{d\xi},\ \beta\right) L_j(\xi,\ \beta)\right]\{U_j(\eta)\} = 0,\ j = 0,\ 1,\ \ldots,\ n-1. \tag{3.29}$$

Since the initial functions are arbitrary, this means that standing in brackets operator's value on by arbitrary function is equal to

zero, and this is possible only in such a case, when operator himself zero (see page 28), i. e.,

$$D\left(\frac{d}{d\xi}, \beta\right) L_j(\xi, \beta) = 0, \quad (j = 0, 1, \ldots, n-1. \tag{3.30}$$

Thus we obtained n of the ordinary differential equations of the n order with constant coefficients for the unknown operator-functions.

Obviously, it is necessary to still dispose of conditions for determining n^2 integration constants (this will be also the operators). In order to obtain them, let us substitute (3.24) in (3.22):

$$U_s(\eta) = \sum_{j=0}^{n-1} \frac{d^s L_j}{d\xi^s}\bigg|_{\xi=0} U_j(\eta), \quad s = 0, 1, \ldots, n-1. \tag{3.31}$$

Hence, again keeping in mind the independence of the initial functions, we obtain

$$\frac{d^s L_j}{d\xi^s}\bigg|_{\xi=0} = \delta_{sj}; \; s, j = 0, 1, \ldots, n-1, \tag{3.32}$$

where δ_{sj} — Kronecker's symbol. But if as the initial functions are accepted expressions (3.23), then instead of (3.31) it will be

$$U_k(\eta) = \sum_{s=0}^{n-1} b_{ks}\left[\sum_{j=0}^{n-1} \frac{d^s L_j}{d\xi^s}\bigg|_{\xi=0} U_j(\eta)\right] = \sum_{j=0}^{n-1}\left[\sum_{s=0}^{n-1} b_{ks}\frac{d^s L_j}{d\xi^s}\right]_{\xi=0} U_j(\eta). \tag{3.33}$$

whence

$$\sum_{s=0}^{n-1} b_{ks} \frac{d^s L_j}{d\xi^s}\bigg|_{\xi=0} = \delta_{jk};\ j,\ k=0,\ 1,\ \ldots,\ n-1. \qquad (3.34)$$

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By integrating now equations (3.30) under initial conditions (3.32) or (3.34), let us find all operators entering in (3.24), and consider that is concealed by form the general solution constructed. As concerns particular solution u_r, about its determination it will go speech in §16.

In many instances of value (3.23) they make sense and are of interest not only on boundary, but also at the points of domain. Then it is expedient somewhat to modify notations. Let us write the basic system of the differential equations:

$$D(\alpha,\ \beta)\{u(\xi,\ \eta)\} = f(\xi,\ \eta),$$

$$u_j(\xi,\ \eta) = \sum_{s=0}^{n-1} b_{js} \frac{\partial^s u}{\partial \xi^s},\ j=0,\ 1,\ 2,\ \ldots,\ n-1. \qquad (3.35)$$

Let us select as the initial functions

$$U_j = U_j(\eta) = u_j(0, \eta), \quad (u_0 = u), \quad j = 0, 1, 2, \ldots, n-1 \qquad (3.36)$$

and seek the general solution in the form

$$\begin{aligned} u(\xi, \eta) &= L_{00}(\xi, \beta) U_0(\eta) + L_{01}(\xi, \beta) U_1(\eta) + \ldots + \\ &+ L_{0, n-1}(\xi, \beta) U_{n-1}(\eta) + U_f, \\ u_1(\xi, \eta) &= L_{10}(\xi, \beta) U_0(\eta) + L_{11}(\xi, \beta) U_1(\eta) + \ldots + \\ &+ L_{1, n-1}(\xi, \beta) U_{n-1}(\eta) + U_{1f}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ u_{n-1}(\xi, \eta) &= L_{n-1, 0}(\xi, \beta) U_0(\eta) + L_{n-1, 1}(\xi, \beta) U_1(\eta) + \ldots + \\ &+ L_{n-1, n-1}(\xi, \beta) U_{n-1}(\eta) + u_{n-1f}. \end{aligned} \qquad (3.37)$$

Here u_f it makes previous sense, and

$$u_{jf} = \sum_{s=0}^{n-1} b_{js} \frac{\partial^s u_{jf}}{\partial \xi^s}. \qquad (3.38)$$

After substituting (3.37) in (3.35), we will obtain the system of the ordinary differential equations

$$D\left(\frac{d}{d\xi}, \beta\right) L_{0k}(\xi, \beta) = 0,$$

$$L_{jk}(\xi, \beta) = \sum_{s=0}^{n-1} b_{js} \frac{d^s L_{0k}(\xi, \beta)}{d\xi^s}, \qquad (3.39)$$

$$k = 0, 1, \ldots, n-1; \; j = 1, 2, \ldots, n-1.$$

while after substituting (3.37) in (3.36) we will obtain the necessary for determining n^2 integration constants initial conditions

$$L_{jk}(0, \beta) = \delta_{jk}; \quad j, k = 0, 1, \ldots, n-1. \tag{3.40}$$

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After performing integration, let us find all operators, and also, therefore, let us construct the general solution. Expressions (3.37) can be named the canonical equations of the method of the initial functions or the canonical form of general solution.

Here we are limited to the problems, stated for a rectangular domain and by the described differential equations in partial derivatives with constant coefficients. If domain rectangular, but the initial equation has variable coefficients (i.e. is solved problem in orthogonal curvilinear coordinates for the domain, limited by coordinate lines), then in many instances of complication concerns only equations (3.30) and (3.39) - this will be ordinary differential

equations with variable coefficients. Examples of such solutions will be examined into §§17 and 20.

§13. Satisfaction to boundary conditions. Resolving equations.

If we the differential operator D (α, β) hyperbolic and in the task in question eat the only initial conditions, then all the functions $u_j(\eta)$ are known and expression (3.24) it gives the ready solution of problem, written in operational form (analogous to (with 3.18)). By realizing operators, let us arrive at the usual form of solution.

More complexly is matter in the case of boundary-value problem, when not all n of conditions are assigned on the initial line; certain number N of conditions is related to opposite boundary of the region (ξ = 1). If as the initial functions are accepted expressions (3.23), then this means that n - N initial functions will be known

$$U_j(\eta)=\sum_{s=0}^{n-1} b_{js}\left(\frac{\partial^s u}{\partial \xi^s}\right)_{\xi=0}=\psi_j(\eta), \quad j=0, 1, \ldots, n-N-1, \quad (3.41)$$

and N of functions U_j (j = n - N, n — N + 1, ..., n — 1) they will remain unknowns. With even n usually N = 0.5n.

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Conditions at edge $\xi = 1$ take the same form

$$u_j(1, \eta) - \sum_{s=0}^{n-1} b_{js}\left(\frac{\partial^s u}{\partial \xi^s}\right)_{\xi=1} = \psi_j(\eta), \quad j=n-N, n-N+1, \ldots, n-1, \tag{3.42}$$

since as initial are selected the functions, which make in this task concrete/specific/actual physical sense. Outside in (3.42) function U_j from general solution (3.37), we will obtain for unknown initial functions system N of the equations, written in the operational form:

$$L_{j0}(1, \beta)U_0(\eta) + L_{j1}(1, \beta)U_1(\eta) + \ldots + L_{j, n-1}(1, \beta)U_{n-1}(\eta) + $$
$$+ u_{n-j}f(1, \eta) = \psi_j(\eta), \quad j = n-N, n-N+1, \ldots, n-1 \tag{3.43}$$

or

$$L_{j, n-N}(1, \beta)U_{n-N}(\eta) + \ldots + L_{j, n-1}(1, \beta)U_{n-1}(\eta) = \Psi_j(\eta), \quad j = n- $$
$$-N, n-N+1, \ldots, n-1, \tag{3.44}$$

where Ψ_j known functions.

Depending on that, in which form are here written the operators, to this system it is possible to give the different interpretations:

1) if the operators are given in the closed form, then this it will be the system of transcendental of ordinary differential equations;

2) if the operators are decomposed in series and moreover are regular, then will be obtained the system of the ordinary differential equations of infinitely high order; if the operators mixed - that is the system of the integrodifferential equations of infinitely high order;

3) if the operators are realized by the formulas of chapters 1 and 2, then this there will be the system of some functional equations (final integrodifferential, difference, differential-difference).

For finding the particular solutions of a heterogeneous system it is necessary, obviously, to use that treatment, which faster leads to the result: if $\Phi_i(\eta)$ polynomials, then p. 2, if $\Phi_i(\eta)$ exponential, trigonometric or hyperbolic functions, then p. 1 or 3. As concerns the determination of the general solution of uniform system (3.44), everywhere subsequently we let us adhere to the first interpretation.

The integration of uniform system (3.44) can be carried out by several methods. Subsequently for this purpose is applied the resolving function $\Phi(\eta)$.

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let us place

$$U_s(\eta)=L_s(1,\beta)\varphi(\eta),\quad s=n-N,\ldots,\ n-1. \qquad (3.45)$$

where $L_s(1,\beta)$ the operator, which will be obtained as a result of expansion of a determinant, which is the cofactor of cell/element $L_{n-1,s}(1,\beta)$ in the determinant of system (3.44). Then will be

$$\sum_{s=n-N}^{n-1} L_{js}(1,\beta)L_s(1,\beta)=0,\quad j=n-N,\ldots,\ n-2 \qquad (3.46)$$

and, therefore, the left sides of all equations (3.44), with the exception of the latter (j = n - 1), they will become zero with any $\varphi(\eta)$. By substituting now (3.45) in this last/latter equation, we will obtain equation for determining the resolving function

$$L\varphi=\left[\sum_{s=n-N}^{n-1} L_{n-1,s}(1,\beta)L_s(1,\beta)\right]\varphi(\eta)=0. \qquad (3.46)$$

Let us name it the resolving equation.

In the overwhelming majority of the real tasks of the theory of elasticity (yes even generally mathematical physics) all operators, entering general solution (3.37), turn out to be regular. But then regular will be operator L.

The resolving equaticn allow/assume three noted above interpretaticn. By adhering to the first of them, let us consider that (3.46) this transcendental ordinary differential equation [1], whereupon with constant coefficients. Its solution logically to seek in the form

$$\varphi(\eta)=e^{k\eta}. \tag{3.47}$$

FCOTNOTE [1]. Examples of other treatments are brought in §23. ENDFOOTNOTE.

Cutside (3.47) in (3.46) and after taking into account (1.192), let us arrive at the transcendental characteristic equation

$$L(k)=0. \tag{3.48}$$

Generally speaking, it has an infinite multitude of complex roots. Therefore

$$\varphi(\eta)=\sum_{s=0}^{\infty}A_s e^{k_s\eta}, \tag{3.49}$$

where A_s arbitrary (complex) constants. This expression is

written for that case, when all roots of equation (3.48) simple. But if be and multiple roots, then in (3.49) it is necessary to introduce the appropriate polynomial factors.

by substituting now (3.49) in (3.45), and then result in (3.37), let us find the unknown functions in the form of series with arbitrary coefficients.

$$u_l(\xi,\eta)=F_l(\xi,\eta)+\sum_{l=0}^{\infty}\left\{\left[\sum_{s=1}^{N}A_{lsl}\varphi_{lsl}(\eta)\right]\Phi_{ll}(\xi)\right\}. \quad (3.50)$$

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Coefficients A_{lsl} must be determined from boundary conditions at edges $\eta=0$ and $\eta=\lambda$:

$$\left.\begin{aligned}u_l(\xi,\eta)=F_l(\xi,0)+\sum_{l=0}^{\infty}\left\{\left[\sum_{s=1}^{N}A_{lsl}\varphi_{lsl}(0)\right]\Phi_{ll}(\xi)\right\}=p_{0l}(\xi).\\ u_l(\xi,\lambda)=F_l(\xi,\lambda)+\sum_{l=0}^{\infty}\left\{\left[\sum_{s=1}^{N}A_{lsl}\varphi_{lsl}(\lambda)\right]\Phi_{ll}(\xi)\right\}=p_{\lambda l}(\xi).\end{aligned}\right\} \quad (3.51)$$

If set of functions $\Phi_{ll}(\xi)$ is orthogonal, then it is possible to find everything A_{lsl}, as a result of which we will obtain precise the solutions to task; but if functions $\Phi_{ll}(\xi)$ are not orthogonal, then solution it will be that which was approximated - it accurately it satisfies equations (3.35) and to conditions at edges $\xi = 0$ and $\xi = 1$, but to approximate-boundary conditions at edges $\eta=0$ and $\eta=\lambda$.

As concerns the determination of the approximate value of coefficients, here possibly application/use of the different methods: minimization of the root-mean-square deviation, collocation (point to satisfaction of boundary conditions), orthogonalization of a series with respect to any set of functions $\{ \rho (x) \}$ (specifically, the application/use of a method of torque/moments), etc. The first method (minimization) must give better/best for the taken into consideration quantity m of the terms of a series of result, but it, as a rule, is conjugate/combined with cumbersome calculations. Collocation with small m usually gives the sharp bursts between the points, where the conditions are satisfied accurately a with large m it leads to the strongly oscillatory functions. To fair results it leads the method of orthogonalization, but only in such a case, when successfully (for the task in question) is selected system $\{ \rho (x) \}$. According to our opinion, most rapidly conducts to target/purpose the following method: let us write conditions (3.51) in the form

$$\left.\begin{aligned} f_{0l}(\xi) = u_l(\xi, 0) - p_{0l}(\xi) = 0, \\ f_{\lambda l}(\xi) = u_l(\xi, \lambda) - p_{\lambda l}(\xi) = 0, \end{aligned}\right\} \tag{3.52}$$

is decomposed functions $f_{0l}(\xi)$ and $f_{\lambda l}(\xi)$ in series according to degrees ξ

$$f_{sl}(\xi) = a_0 + a_1\xi + a_2\xi^2 + \ldots, \tag{3.53}$$

either $(1 - \xi)$

$$f_{sj}(\xi) = b_0 + b_1(1-\xi) + b_2(1-\xi)^2 + \ldots \quad (3.54)$$

and let us find $A_{ls j}$, by equalizing zero several first coefficients of series (3.53) or (3.54). Examples of the application/use of this methcd are given in chapter 5.

In conclusion let us note that if the task possesses explicit symmetry, then the initial line expedient tc chcose on the axis of symmetry.

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§14. Determination of the roots of transcendental characteristic equaticns.

One of the bulky stages in the solution to specific problems is the determination of the roots of transcendental characteristic equation (3.48). This unwieldiness is connected with the fact that equation (3.48) has infinitely many complex roots. In the literature are sufficiently widely illuminated both methods of the calculation of these roots and the general theory cf transcendental equations.

Therefore here we will pause briefly only at some practical questions. Conventional is the preliminary determination of approximate values of k graphic or tabular method, and then the refinement of roots according to Newton's method. The research of approximate values is facilitated, if it is possible to obtain asymptotic formulas for roots. Are most common/general/total results on this question are given, apparently, in [33], chapter 3.

However, in concrete/specific/actual cases it is possible to find the iterative methods, which ensure more rapid than from Newton's method, the convergence of the process successive approximation and the more light/lung determination of asymptotic roots. As an example let us examine the equation

$$\sin z = \nu z, \quad \nu > 0. \tag{3.55}$$

Obviously, if z is a root of this equation, then $-z$, $\bar{z}$ and $-\bar{z}$ also will be roots. Therefore, after assuming

$$z = x + iy, \tag{3.56}$$

we can count x, y > 0. Let us divide in (3.55) in natural and apparent/imaginary part

$$\sin x \operatorname{ch} y = \nu x, \tag{3.57}$$

$$\cos x \operatorname{sh} y = \nu y. \tag{3.58}$$

In order to find real roots, let us place y = 0. Then

$$\sin x = \nu x, \tag{3.59}$$

a curve/graphs (Fig. 3) it shows, that with $\nu > 1$ there are no real roots, and with $\nu < 1$ will be one or several real roots.

Let now x = 0 and, consequently,

$$\operatorname{sh} y = \nu y. \tag{3.60}$$

From curve/graph (Fig. 4) it is evident that with $\nu < 1$ there are no pure imaginary roots, and with $\nu > 1$ there will be one such root.

Further let us present (3.55) in the form of the series

$$(1-\nu)z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots = 0, \tag{3.61}$$

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Hence it is apparent that with $\nu \neq 1$ equation (3.55) has idle time, and with $\nu = 1$ the triple root z = 0.

Let us pass to the determination of complex roots. From (3.57) and (3.58) it follows that

$$x^2 = \frac{\operatorname{ch}^2 y}{\nu^2} - y^2 \operatorname{cth}^2 y. \tag{3.62}$$

With sufficiently large $y\frac{\operatorname{ch} y}{v} \gg y \operatorname{cth} y$, and therefore the approximate (asymptotic) value x will be

$$x_0 = \frac{\operatorname{ch} y_0}{v}. \tag{3.63}$$

Introducing this expression in (3.57), we obtain

$$\sin x_0 = 1, \tag{3.64}$$

whence

$$x_0 = \frac{\pi}{2}(4s+1), \quad s = 0, 1, 2, \ldots. \tag{3.65}$$

Then from (3.62) the precise value

$$x_s = \frac{\pi}{2}(4s+1) - \delta_s, \quad s = 0, 1, 2, \ldots, \tag{3.66}$$

where δ_s - generally speaking, small number.

The minimum value s depends on value v. Let us estimate s_{min}. From (3.63) and (3.65)

$$\operatorname{ch} y_0 = v\frac{\pi}{2}(4s+1). \tag{3.67}$$

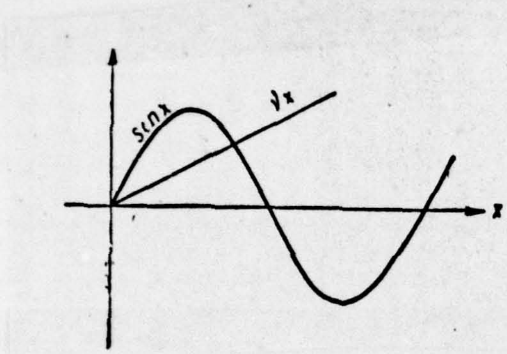

Fig. 3.

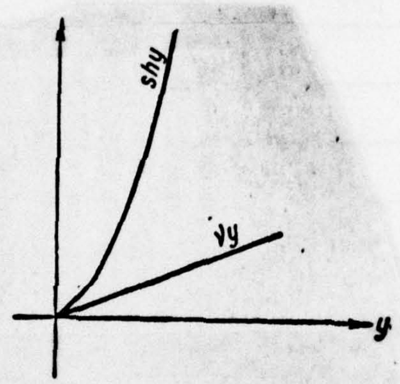

Fig. 4.

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Since must be $y_0 \geqslant 1$, hence it follows that

$$\left.\begin{aligned} s_{min} &= 0, \text{ если}^{(1)}\ \nu > \frac{2}{\pi}, \\ s_{min} &= 1, \text{ если}^{(1)}\ \frac{2}{\pi} > \nu > \frac{2}{5\pi}, \\ s_{min} &= 2, \text{ если}^{(1)}\ \frac{2}{5\pi} > \nu > \frac{2}{9\pi} \end{aligned}\right\} \quad (3.68)$$

Key: (1). if.

and so forth.

From (3.66) it escape/ensues

$$\sin x_s = \cos \delta_s, \quad \cos x_s = \sin \delta_s. \tag{3.69}$$

Then from (3.57)

$$y_s = \operatorname{arch}\left(\frac{\nu x_s}{\cos \delta_s}\right), \tag{3.70}$$

a from (3.58)

$$\delta_s = \arcsin\left(\frac{\nu y_s}{\operatorname{sh} y_s}\right). \tag{3.71}$$

After substituting (3.66) in (3.70), and result in (3.71), we will obtain

$$\delta_s = \arcsin\left\{\frac{\nu \operatorname{arch}\left[\frac{\nu}{\cos \delta_s}\left(\frac{\pi}{2}(4s+1) - \delta_s\right)\right]}{\sqrt{\left[\frac{\nu}{\cos \delta_s}\left(\frac{\pi}{2}(4s+1) - \delta_s\right)\right]^2 - 1}}\right\}. \tag{3.72}$$

Iterational process due to this formula rapidly leads to the sufficiently precise values of roots (see §24). Analogous formulas can be obtained also for other transcendental equations. But if obtaining a similar formula turns out to be difficult, then is applied Newton's method. In this case the convergence of process can be somewhat accelerated.

Let us write equation (3.55) in the form

$$f(z) = \sin z - \nu z = 0 \tag{3.73}$$

and designate by z_0 the initial approximate value of root, but by z_1 - refined. Then by Newton's formula

$$\Delta z = z_1 - z_0 = -\frac{f(z_0)}{f'(z_0)} = \frac{\sin z_0 - \nu z_0}{\nu - \cos z_0}. \quad (3.74)$$

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This formula can be obtained by the method, which slightly differs from conventional. Let

$$\sin(z_0 + \Delta z) - \nu(z_0 + \Delta z) = 0, \quad (3.75)$$

i. e.,

$$\sin z_0 \cdot \cos \Delta z + \cos z_0 \sin \Delta z - \nu(z_0 + \Delta z) = 0. \quad (3.76)$$

After assuming here

$$\cos \Delta z = 1, \quad \sin \Delta z = \Delta z, \quad (3.77)$$

let us arrive at (3.74). But if we take

$$\cos \Delta z = 1 - \frac{(\Delta z)^2}{2}, \quad \sin \Delta z = \Delta z, \quad (3.78)$$

that more precise, than for (3.74), value Δz will be located as smaller in the module/modulus of the root of the quadratic equation

$$\frac{\sin z_0}{2}(\Delta z)^2 + (\nu - \cos z_0)\Delta z - (\sin z_0 - \nu z_0) = 0. \quad (3.79)$$

§15. Solution of V. Z. Vlasov.

Let us take the second of three noted on page 131 treatments of system (3.44) we will be bounded in the expansions of the operators by several first members. Then (3.44) it is converted into the system of the ordinary differential equations of the final order. By its it is possible to integrate usual methods, including the introduction of the resolving function. By accepting as before (3.45), let us arrive at resolving equation (3.46), which now will take the form

$$(\beta^m + C_1\beta^{m-1} + \ldots + C_{m-1}\beta + C_m)\varphi(\eta) = 0. \qquad (3.80)$$

After integrating it, let us arrange/locate m the constants, which will permit to satisfy m conditions at edges $\eta = 0$ and $\eta = \lambda$. As a result is obtained the solution, which it satisfies only approximately both the initial differential equations and all boundary conditions. Then in this case is decreased the volume of

computational work.

This method coincides actually with the fact, that was proposed to V. Z. Vlasov for the solution of the three-dimensional problem of elasticity theory. Actually, searching the solution to equations (3.35) in the form (analogous to (with 0.26))

$$u_i(\xi, \eta) = U_i + \xi\left(\frac{\partial u_i}{\partial \xi}\right)_{\xi=0} + \frac{\xi^2}{2!}\left(\frac{\partial^2 u_i}{\partial \xi^2}\right)_{\xi=0} + \dots \qquad (3.81)$$

and entering further in the manner that it was described on page 15, we will arrive at thereby (truncated) operators and resolving equation (3.80).

This method of solution is used to axisymmetric thermoelastic task in the article [23]. To the described method expediently to resort when the construction of general solution (in essence the integration of system (3.44)) causes considerable difficulties.

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Chapter 4.

SOME GENERAL SOLUTIONS.

In chapter 4 are given examples of the construction of general solutions for some two-dimensional tasks of the applied theory of elasticity in accordance with the plan/layout, presented in §12. General solution (3.37) contains the uniform part, expressed by the terms of form $L_{l_0}(\xi,\beta)u_0(\eta)$, and a heterogeneous part (component u_{ll}), connected with particular solution u_l nonhomogeneous differential equation (3.35). First will be examined a heterogeneous part of the general solution.

§16. On the particular solution to nonhomogeneous differential equation.

Let it is required to find this solution u_l the equation

$$D(\alpha,\beta)u(\xi,\eta)=f(\xi,\eta), \tag{4.1}$$

(where the differential operator D (α, β) it is determined by formula

(3.21)), which on the initial line satisfies conditions (3.25).

If we write formally

$$u_1(\xi,\eta)=\frac{1}{D(\alpha,\beta)}f(\xi,\eta). \qquad (4.2)$$

that question will be reduced to operator's realization $1/D(\alpha,\beta)$ above the function $f(\xi,\eta)$. For this purpose it is necessary to use p. 4 §10. As it is there noted, singular operator $1/D(\alpha,\beta)$ has several forms of realization. It is necessary to select that (if it exists), that ensures satisfaction of conditions (3.25). Moreover (since expression (4.2) it is obtained formally) necessary to check, really/actually the obtained function $u_1(\xi,\eta)$ is the solution to equation (4.1). Let us consider several examples.

1. The equation of Poisson in Cartesian coordinates is such:

$$\Delta u=\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}=f(\xi,\eta). \qquad (4.3)$$

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Here

$$D(\alpha,\beta)=\alpha^2+\beta^2 \qquad (4.4)$$

and, according to (2.162),

$$\boxed{u_1(\xi,\eta)=\frac{1}{2i}\int_0^{\xi}dt\int_{\eta-i(\xi-t)}^{\eta+i(\xi-t)}f(t,\zeta)\,d\zeta.} \qquad (4.5)$$

Differentiating, we find

$$\left.\begin{aligned}
\alpha u_f &= \frac{1}{2}\int_0^{\xi}\{f[t, \eta + i(\xi - t)] + f[t, \eta - i(\xi - t)]\}\, dt,\\
\alpha^2 u_f &= \frac{i}{2}\int_0^{\xi}\left\{\left.\frac{\partial f(t, v)}{\partial v}\right|_{v=\eta+i(\xi-t)} - \left.\frac{\partial f(t, v)}{\partial v}\right|_{v=\eta-i(\xi-t)}\right\} dt + f(\xi, \eta),\\
\beta u_f &= \frac{1}{2i}\int_0^{\xi}\{f[t, \eta + i(\xi - t)] - f[t, \eta - i(\xi - t)]\}\, dt,\\
\beta^2 u_f &= \frac{1}{2i}\int_0^{\xi}\left\{\left.\frac{\partial f(t, v)}{\partial v}\right|_{v=\eta+i(\xi-t)} - \left.\frac{\partial f(t, v)}{\partial v}\right|_{v=\eta-i(\xi-t)}\right\} dt.
\end{aligned}\right\} \quad (4.6)$$

Hence it is apparent that expression (4.5) satisfies both equation (4.3) and to the conditions

$$u_f(0, \eta) = \left.\frac{\partial u_f}{\partial \xi}\right|_{\xi=0} = 0. \quad (4.7)$$

2. Equation

$$\frac{\partial^2 u}{\partial \xi^2} + \lambda^2 \frac{\partial^2 u}{\partial \eta^2} = f(\xi, \eta). \quad (4.8)$$

Comparing this equation with (4.3), we see that for obtaining u_f is sufficient in (4.5) to replace i on iλ

$$u_f(\xi, \eta) = \frac{1}{2i\lambda}\int_0^{\xi} dt \int_{\eta - i\lambda(\xi - t)}^{\eta + i\lambda(\xi - t)} f(t, \zeta)\, d\zeta. \quad (4.9)$$

3. Equation

$$\frac{\partial^4 u}{\partial \xi^4} + 2\lambda^2 \frac{\partial^4 u}{\partial \xi^2 \partial \eta^2} + \lambda^4 \frac{\partial^4 u}{\partial \eta^4} = f(\xi, \eta). \qquad (4.10)$$

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Here

$$u_1(\xi, \eta) = \frac{1}{(\alpha^2 + \lambda^2\beta^2)^2} f(\xi, \eta) = \frac{1}{\alpha^2 + \lambda^2\beta^2} \left\{ \frac{1}{\alpha^2 + \lambda^2\beta^2} f(\xi, \eta) \right\}. \qquad (4.11)$$

Therefore, twice by applying formula (4.9), we will obtain

$$\boxed{u_1(\xi, \eta) = -\frac{1}{4\lambda^2} \int_0^{\xi} d\zeta \int_{\eta - i\lambda(\xi - \zeta)}^{\eta + i\lambda(\xi - \zeta)} dt \int_0^{\zeta} d\zeta' \int_{t - i\lambda(\zeta - \zeta')}^{t + i\lambda(\zeta - \zeta')} f(\zeta', t')\, dt'.} \qquad (4.12)$$

Elementary, but sufficiently bulky checking shows that this expression it satisfies both equation (4.10) and to the conditions

$$\left.\frac{\partial^{s+n} u_1}{\partial \xi^s \partial \eta^n}\right|_{\xi=0} = 0, \quad 0 < s+n < 3.$$

§17. Equation of Poisson.

1. Cartesian coordinates

Let the function $u(\xi, \eta)$ in rectangle (Fig. 2) satisfy the equation

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = f(\xi, \eta). \tag{4.13}$$

Let us take the initial functions of type (3.22)

$$U_0(\eta) = u(0, \eta), \quad U_1(\eta) = \left(\frac{\partial u}{\partial \xi}\right)_{\xi=0} \tag{4.14}$$

and in accordance with (3.37) seek the general solution in the form

$$u(\xi, \eta) = L_{00}(\xi, \beta) U_0(\eta) + L_{01}(\xi, \beta) U_1(\eta) + u_f, \tag{4.15}$$

$$\frac{\partial u}{\partial \xi} = u_1(\xi, \eta) = L_{10}(\xi, \beta) U_0(\eta) + L_{11}(\xi, \beta) U_1(\eta) + u_{1f}. \tag{4.16}$$

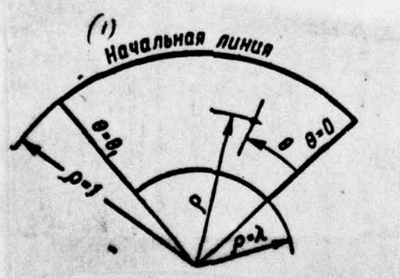

Fig. 5.

KEY: (1). Initial line

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After selecting as u_1 expression (4.5) and after substituting (4.15) in (4.13), let us arrive at equations (3.30)

$$\frac{d^2 L_{0j}}{d\xi^2} + \beta^2 L_{0j} = 0, \quad j = 0,1. \tag{4.17}$$

Hence

$$L_{00} = A_0 \cos \xi\beta + B_0 \sin \xi\beta, \quad L_{01} = A_1 \cos \xi\beta + B_1 \sin \xi\beta. \tag{4.18}$$

According to (3.32)

$$L_{00}(0, \beta) = 1, \quad \left.\frac{dL_{00}}{d\xi}\right|_{\xi=0} = 0, \quad L_{01}(0, \beta) = 0, \quad \left.\frac{dL_{01}}{d\xi}\right|_{\xi=0} = 1. \tag{4.19}$$

Therefore

$$L_{00}=\cos\xi\beta,\quad L_{01}=\frac{\sin\xi\beta}{\beta}. \tag{4.20}$$

Differentiating (4.15) with respect to ξ and comparing result with (4.16), we find

$$L_{10}=-\beta\sin\xi\beta,\quad L_{11}=\cos\xi\beta. \tag{4.21}$$

Thus,

$$u(\xi,\eta)=\cos\xi\beta U_0(\eta)+\frac{\sin\xi\beta}{\beta}U_1(\eta)+\frac{1}{2i}\int_0^{\xi}dt\int_{\eta-i(\xi-t)}^{\eta+i(\xi-t)}f(t,\zeta)\,d\zeta. \tag{4.22}$$

2. Polar coordinates (initial line is curvilinear).

Function u (ρ,θ) in region $\lambda\leqslant\rho\leqslant 1$, $0\leqslant\theta\leqslant\theta_1$ (Fig. 5) satisfies the equation

$$\Delta u=\frac{\partial^2 u}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\theta^2}=f(\rho,\theta). \tag{4.23}$$

Here ρ is a dimensionless radius (radius, referred to an outside

radius). Let us take for the initial line external arc ($\rho = 1$), and for the initial functions

$$U_0 = U_0(\theta) = u(1, \theta), \quad U_1 = U_1(\theta) = \left.\frac{\partial u}{\partial \rho}\right|_{\rho=1} \tag{4.24}$$

let us find out the general solution in the form

$$U(\rho, \theta) = L_0(\rho, \beta)\, U_0(\theta) + L_1(\rho, \beta)\, U_1(\theta) + u_j^* . \tag{4.25}$$

where markedly

$$\beta = \frac{d}{d\theta}. \tag{4.26}$$

FOOTNOTE [1]. In this simple case there is no need to retain the dual indexing of the operators. ENDFOOTNOTE.

Substituting (4.25) in (4.23) (see (3.29)), we will obtain

$$\frac{d^2 L_j}{d\rho^2} + \frac{1}{\rho}\frac{dL_j}{d\rho} + \frac{\beta^2}{\rho^2} L_j = 0, \quad j = 0, 1. \tag{4.27}$$

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Hence

$$L_j = A_j \rho^{i\beta} + B_j \rho^{-i\beta} = A_j e^{i \ln \rho \cdot \beta} + B_j e^{-i \ln \rho \cdot \beta}. \tag{4.28}$$

From conditions (see (3.32)) with $\rho = 1$

$$L_0 = 1, \quad \frac{dL_0}{d\varrho} = 0, \quad L_1 = 0, \quad \frac{dL_1}{d\varrho} = 1 \tag{4.29}$$

we find

$$A_0 = B_0 = \frac{1}{2}, \quad A_1 = -B_1 = \frac{1}{2\beta i}. \tag{4.30}$$

Consequently,

$$L_0 = \cos(\beta \ln \varrho), \quad L_1 = \frac{\sin(\beta \ln \varrho)}{\beta} \tag{4.31}$$

and

$$u(\varrho, \theta) = \cos(\beta \ln \varrho) U_0(\theta) + \frac{\sin(\beta \ln \varrho)}{\beta} U_1(\theta) + u_f. \tag{4.32}$$

As concerns particular solution u_f it it is possible to obtain and directly from (4.23), but simpler to use given in the following point/item formula (4.43). Set/assuming in it

$$\xi = \ln \varrho, \quad t = \ln \tau \tag{4.33}$$

and taking into account (4.41), we obtain

$$u_f = \frac{1}{2i} \int_1^{\varrho} \tau d\tau \int_{\theta - i \ln \frac{\varrho}{\tau}}^{\theta + i \ln \frac{\varrho}{\tau}} f(\tau, \zeta) \, d\zeta. \tag{4.34}$$

It is not difficult to ascertain that this expression satisfies equation (4.23) and to the conditions

$$u_f(1,\theta) = \frac{\partial u_f}{\partial \varrho}\bigg|_{\varrho=1} = 0. \qquad (4.35)$$

In accordance with (1.238) and (1.246) the general solution can be written in the form

$$u(\varrho,\theta) = \mathrm{Re}\,\{U_0(\theta + i\ln\varrho)\} + \frac{1}{2i}\int_{\theta - i\ln\varrho}^{\theta + i\ln\varrho} U_1(\zeta)\,d\zeta + u_f. \qquad (4.36)$$

3. Given polar coordinates. Conversion of independent variable.

$$\xi = \ln\varrho, \quad \varrho = e^{\xi} \qquad (4.37)$$

it translate/transfers the region, depicted on Fig. 5, into the rectangle of Fig. 6.

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In this case

$$\frac{\partial u}{\partial \varrho} = \frac{1}{\varrho}\frac{\partial u}{\partial \xi}, \quad \frac{\partial^2 u}{\partial \varrho^2} = \frac{1}{\varrho^2}\left(\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial u}{\partial \xi}\right) \qquad (4.38)$$

and Laplace's operator (4.23) will pass in

$$\Delta u = e^{-2\xi}\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \theta^2}\right). \tag{4.39}$$

Therefore instead of (4.23) let us have

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \theta^2} = f_1(\xi, \theta), \tag{4.40}$$

where

$$f_1(\xi, \theta) = e^{2\xi} f(e^{\xi}, \theta) = \varrho^2 f(\varrho, \theta). \tag{4.41}$$

Equation (4.40) coincides with (4.13). Consequently, accordingly (4.22)

$$u(\xi, \theta) = \cos \xi\beta\, U_0(\theta) + \frac{\sin \xi\beta}{\beta} U_1(\theta) + u_f, \tag{4.42}$$

where

$$u_f = \frac{1}{2i} \int_0^{\xi} dt \int_{\theta - i(\xi - t)}^{\theta + i(\xi - t)} f_1(t, \zeta)\, d\zeta, \tag{4.43}$$

a

$$U_0 = U_0(\theta) = u(0, \theta), \quad U_1 = U_1(\theta) = \left.\frac{\partial u}{\partial \xi}\right|_{\xi = 0}. \tag{4.44}$$

The general solution can be written also in the form

$$u(\xi, \theta) = \operatorname{Re}\{U_0(\theta + i\xi)\} + \frac{1}{2i} \int_{\theta - i\xi}^{\theta + i\xi} U_1(\zeta)\, d\zeta + u_f. \tag{4.45}$$

Expression (4.43) satisfies equation (4.40) and to the conditions

$$u_1(0,\theta)=\left.\frac{\partial u_1}{\partial \xi}\right|_{\xi=0}=0. \qquad (4.46)$$

Let us select now the initial line on the axis of abscissas (Fig. 7). The initial functions they will be

$$U_0=U_0(\xi)=u(\xi,0), \quad U_1=U_1(\xi)=\left.\frac{\partial u}{\partial \theta}\right|_{\theta=0}. \qquad (4.47)$$

Since equation (4.40) is symmetrical relative to variables ξ and θ, for obtaining general solution in this case it suffices to interchange the position ξ and θ in (4.45), (4.42) and (4.43) and to replace β by

$$\alpha=\frac{d}{d\xi}=\frac{d}{d\ln\varrho}. \qquad (4.48)$$

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Thus will be

$$u(\xi,\theta)=\cos\theta\alpha U_0(\xi)+\frac{\sin\theta\alpha}{\alpha}U_1(\xi)+u_1, \qquad (4.49)$$

or

$$u(\xi,\theta)=\mathrm{Re}\{U_0(\xi+i\theta)\}+\frac{1}{2i}\int\limits_{\xi-i\theta}^{\xi+i\theta}U_1(\zeta)\,d\zeta+u_1 \qquad (4.50)$$

and

$$u_1 = \frac{1}{2i}\int_0^\theta dt \int_{\xi - i(\theta - t)}^{\xi + i(\theta - t)} f_1(\zeta, t)\, d\zeta. \qquad (4.51)$$

The last/latter expression satisfies equation (4.40) and to the conditions

$$u_1(\xi, 0) = \left.\frac{\partial u_1}{\partial \theta}\right|_{\theta = 0} = 0. \qquad (4.52)$$

4. Polar coordinates (initial line is rectilinear).

In p. 2 was obtained general solution when the initial line was the external arc boundary of the region (see Fig. 5). If we for the initial line take one of the radial boundaries (Fig. 8), then the corresponding general solution can be easily obtained from formula (4.50)

$$u(\varrho, \theta) = \mathrm{Re}\,\{U_0(\ln \varrho + i\theta)\} + \frac{1}{2i}\int_{\ln \varrho - i\theta}^{\ln \varrho + i\theta} U_1(\zeta)\, d\zeta + u_1. \qquad (4.53)$$

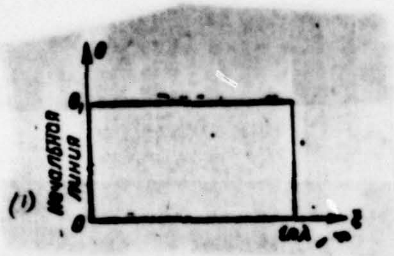

Fig. 6.

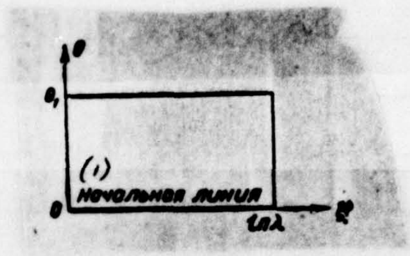

Fig. 7.

Fig. 6.

Key: (1). Initial line.

Fig. 7.

Key: (1). Initial line.

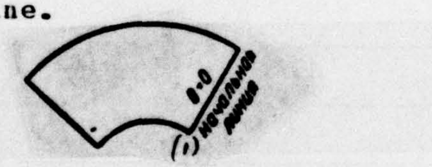

Fig. 8.

Fig. 8.

Key: (1). Initial line.

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For u_l, after producing in (4.51) the replacement

$$\xi = \ln \rho, \quad \zeta = \ln \tau, \tag{4.54}$$

it will have

$$u_l = \frac{1}{2l}\int_0^\theta dt \int_{\rho e^{-l(\theta-t)}}^{\rho e^{l(\theta-t)}} \tau f(\tau, t)\, d\tau. \tag{4.55}$$

The last/latter expression satisfies equation (4.23) and to the conditions

$$u_l(\rho, 0) = \frac{\partial u_l}{\partial \theta}\bigg|_{\theta=0} = 0. \tag{4.56}$$

§18. Two-dimensional problem of elasticity theory. Thermoelastic task.

Let us consider the rectangular plate (Fig. 9) of thickness δ, which experience/tests state of plane stress of the action of loads on outline/contour and the vertical volume forces q (kg/cm³). To Fig. 9 are shown the positive directions of the appearing in plate

voltage/stresses $\sigma_x, \sigma_y, \tau_{xy}=\tau_{yx}$ and displacement/movements u and v. Are introduced the dimensionless coordinates

$$\xi=\frac{x}{a},\quad \eta=\frac{y}{a},\quad \left(\lambda=\frac{b}{a}\right) \tag{4.57}$$

and the dimensionless static and geometric values

$$\bar{\sigma}_x=\frac{\sigma_x}{G},\quad \bar{\sigma}_y=\frac{\sigma_y}{G},\ \bar{\tau}_{xy}=\frac{\tau_{xy}}{G},\quad U=\frac{u}{a},\quad V=\frac{v}{a},\quad \bar{q}=\frac{a}{G}q. \tag{4.58}$$

Complete system of equations for state of plane stress takes the form

$$\frac{\partial\bar{\sigma}_x}{\partial\xi}+\frac{\partial\bar{\tau}_{xy}}{\partial\eta}=0,\quad \frac{\partial\bar{\tau}_{xy}}{\partial\xi}+\frac{\partial\bar{\sigma}_y}{\partial\eta}=\bar{q},$$

$$\frac{\partial U}{\partial\eta}+\frac{\partial V}{\partial\xi}=\bar{\tau}_{xy},\quad \frac{\partial U}{\partial\xi}-\frac{\partial V}{\partial\eta}=\frac{\bar{\sigma}_x-\bar{\sigma}_y}{2},$$

$$\frac{\partial U}{\partial\xi}=\frac{\bar{\sigma}_x-\mu\bar{\sigma}_y}{2(1+\mu)}. \tag{4.59}$$

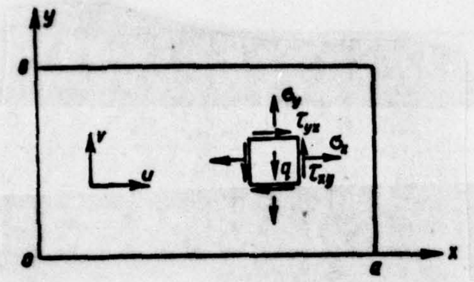

Fig. 9.

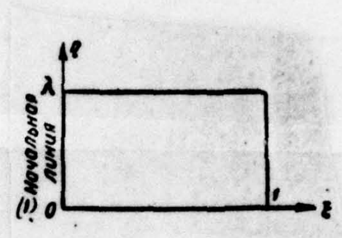

Fig. 10.

Fig. 10.

Key: (1). Initial line.

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Is selected the initial line at edge $\xi = 0$ (Fig. 10). At the points cf this face they make physical sense of value

$$U, \ V, \ \sigma=\bar{\sigma}_x, \ \tau=\bar{\tau}_{xy}. \tag{4.60}$$

Therefore the initial functions they will be

$$\left.\begin{array}{ll} U_0=U_0(\eta)=U(0,\eta), & V_0=V_0(\eta)=V(0,\eta), \\ \sigma_0=\sigma_0(\eta)=\sigma(0,\eta), & \tau_0=\tau_0(\eta)=\tau(0,\eta), \end{array}\right\} \tag{4.61}$$

a the general solution in canonical form (3.37) it will be written as

follows:

$$\left.\begin{aligned} U(\xi,\eta) &= L_{uu}U_0 + L_{uv}V_0 + L_{u\sigma}\sigma_0 + L_{u\tau}\tau_0 + U_q, \\ V(\xi,\eta) &= L_{vu}U_0 + L_{vv}V_0 + L_{v\sigma}\sigma_0 + L_{v\tau}\tau_0 + V_q, \\ \sigma(\xi,\eta) &= L_{\sigma u}U_0 + L_{\sigma v}V_0 + L_{\sigma\sigma}\sigma_0 + L_{\sigma\tau} + \sigma_q, \\ \tau(\xi,\eta) &= L_{\tau u}U_0 + L_{\tau v}V_0 + L_{\tau\sigma}\sigma_0 + L_{\tau\tau}\tau_0 + \tau_q, \\ \bar{\sigma}_y(\xi,\eta) &= L_{\sigma_y u}U_0 + L_{\sigma_y v}V_0 + L_{\sigma_y \sigma}\sigma_0 + L_{\sigma_y \tau}\tau_0 + \sigma_{yq}. \end{aligned}\right\} \quad (4.62)$$

Here U_q, V_q and so forth are quotient of the solution of system (4.59), whereupon such, that with $\xi = 0$

$$U_q = V_q = \sigma_q = \tau_q = \sigma_{yq} = 0. \quad (4.63)$$

We will obtain first this solution. For a brevity let us write system (4.59) in the symbolic form:

$$\alpha\sigma + \beta\tau = 0, \quad (4.64)$$

$$\alpha\tau + \beta\bar{\sigma}_y = q, \quad (4.65)$$

$$\tau = \beta U + \alpha V, \quad (4.66)$$

$$\alpha U - \beta V = \frac{\sigma - \bar{\sigma}_y}{2}, \quad (4.67)$$

$$\alpha U = \frac{\sigma - \mu\bar{\sigma}_y}{2(1+\mu)}. \quad (4.68)$$

From (4.67)

$$\bar{\sigma}_y = \sigma - 2\alpha U + 2\beta V. \quad (4.69)$$

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Let us substitute this expression in (4.65) and (4.68)

$$\sigma = -\frac{2}{1-\mu}(\alpha U + \mu\beta V). \qquad (4.70)$$

$$\beta\sigma + \alpha\tau - 2\alpha\beta U + 2\beta^2 V = \bar{q}. \qquad (4.71)$$

Cutside now expressions for σ and τ from (4.70) and (4.66) in (4.64) and (4.71), we will obtain

$$[2\alpha^2 + (1-\mu)\beta^2]U + (1+\mu)\alpha\beta V = 0. \qquad (4.72)$$

$$(1+\mu)\alpha\beta U + [(1-\mu)\alpha^2 + 2\beta^2]V = \bar{q}(1-\mu). \qquad (4.73)$$

Set/assuming

$$U = -(1+\mu)\alpha\beta\varphi_q(\xi,\eta), \quad V = [2\alpha^2 + (1-\mu)\beta^2]\varphi_q(\xi,\eta), \qquad (4.74)$$

we turn (4.72) into identity, and from (4.73) we find

$$(\alpha^2 + \beta^2)^2\varphi_q = \frac{\bar{q}}{2}. \qquad (4.75)$$

After using now (4.11) and (4.12), we will obtain

$$\varphi_q = -\frac{1}{8}\int_0^{\zeta} d\zeta \int_{\eta - i(\xi-\zeta)}^{\eta + i(\xi-\zeta)} dt \int_0^{\zeta} d\zeta' \int_{t - i(\zeta-\zeta')}^{t + i(\zeta-\zeta')} \bar{q}(\zeta', t')\,dt'. \qquad (4.76)$$

By knowing φ_q, by formulas (4.74), (4.66) and (4.69) it is possible to find $U_q, V_q, \sigma_q, \tau_q$ and $\bar{\sigma}_{\mu q}$. If we fulfill this in general form, then for all functions we will obtain expressions of the type

$$A\int_0^{\zeta} F(\xi,\eta,\zeta)\,d\xi. \qquad (4.77)$$

so that condition (4.63) they will turn out to be those which were carried out. In particular with $\overline{q}$ = const it will be

$$\varphi_q = \frac{qa\xi^4}{48G}, \quad V_q = \frac{\overline{q}\xi^2}{2}, \quad \tau_q = \overline{q}\xi, \qquad U_q = \sigma_q = \overline{\sigma}_{yq} = 0. \tag{4.78}$$

Let us note that the determined by homogeneous equation (4.75) function corresponds to that introduced P. M. Varvak [7] to the function of displacement/movements for two-dimensional problem.

Let us pass to the uniform part of the general solution. One should expressions (4.62) substitute into equations (4.59) or, which is the same, in (4.64) - (4.68). However, since is carried out the already partial separation of functions, let us substitute (4.62) equation (4.72), (4.73), (4.70), (4.66) and (4.69). This will reduce to the following equations:

$$2\frac{d^2L_{uj}}{d\xi^2} + (1-\mu)\beta^2 L_{uj} + (1+\mu)\beta\frac{dL_{vj}}{d\xi} = 0, \tag{4.79}$$

$$(1+\mu)\beta\frac{dL_{uj}}{d\xi} + (1-\mu)\frac{d^2L_{vj}}{d\xi^2} + 2\beta^2 L_{vj} = 0, \tag{4.80}$$

$$\left.\begin{aligned} L_{\sigma j} &= \frac{2}{1-\mu}\frac{dL_{uj}}{d\xi} + \frac{2\mu}{1-\mu}\beta L_{vj},\\ L_{\tau j} &= \beta L_{uj} + \frac{dL_{vj}}{d\xi},\\ L_{\sigma_y j} &= L_{\sigma j} - 2\frac{dL_{uj}}{d\xi} + 2\beta L_{vj}, \quad j = u, v, \sigma, \tau. \end{aligned}\right\} \tag{4.81}$$

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Integrating this system under the conditions (see (3.34))

$$L_{sj}(0, \beta) = \delta_{sj}, \quad s, j = u, v, \sigma, \tau, \tag{4.82}$$

we will obtain (bulky, but elementary conversions we let us lower):

$$
\left.
\begin{aligned}
&L_{uu} = L_{\sigma\sigma} = \cos\xi\beta + \frac{1+\mu}{2}\xi\beta\sin\xi\beta,\\
&L_{uv} = L_{\tau\sigma} = \frac{1-\mu}{2}\sin\xi\beta - \frac{1+\mu}{2}\xi\beta\cos\xi\beta,\\
&L_{u\tau} = L_{v\sigma} = -\frac{1+\mu}{4}\xi\sin\xi\beta,\\
&L_{vv} = L_{\tau\tau} = \cos\xi\beta - \frac{1+\mu}{2}\xi\beta\sin\xi\beta,\\
&L_{\sigma v} = L_{\tau u} = (1+\mu)\xi\beta^2\sin\xi\beta,\\
&L_{vu} = L_{\sigma\tau} = -\frac{1-\mu}{2}\sin\xi\beta - \frac{1+\mu}{2}\xi\beta\cos\xi\beta,\\
&L_{u\sigma} = \frac{3-\mu}{4}\frac{\sin\xi\beta}{\beta} - \frac{1+\mu}{4}\xi\cos\xi\beta,\\
&L_{v\tau} = \frac{3-\mu}{4}\frac{\sin\xi\beta}{\beta} + \frac{1+\mu}{4}\xi\cos\xi\beta,\\
&L_{\sigma u} = -(1+\mu)\beta\sin\xi\beta + (1+\mu)\xi\beta^2\cos\xi\beta,\\
&L_{\tau v} = -(1+\mu)\beta\sin\xi\beta - (1+\mu)\xi\beta^2\cos\xi\beta,\\
&L_{\sigma_y u} = -(1+\mu)\beta\sin\xi\beta - (1+\mu)\xi\beta^2\cos\xi\beta,\\
&L_{\sigma_y v} = 2(1+\mu)\beta\cos\xi\beta - (1+\mu)\xi\beta^2\sin\xi\beta,\\
&L_{\sigma_y \sigma} = \mu\cos\xi\beta - \frac{1+\mu}{2}\xi\beta\sin\xi\beta,\\
&L_{\sigma_y \tau} = \frac{3+\mu}{2}\sin\xi\beta + \frac{1+\mu}{2}\xi\beta\cos\xi\beta.
\end{aligned}
\right\} \qquad (4.83)
$$

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Thus, the general solution for state of plane stress is constructed: all the displacement/movements and voltage/stresses are expressed through the initial functions. For example, according to (4.58), (4.62), (4.74) and (4.83) for horizontal displacement/movements let us have

$$u(\xi,\eta)=a\left\{\left(\cos\xi\beta+\frac{1+\mu}{2}\xi\beta\sin\xi\beta\right)U_0(\eta)+\left(\frac{1+\mu}{2}\sin\xi\beta-\right.\right.$$
$$-\frac{1+\mu}{2}\xi\beta\cos\xi\beta\Big)V_0(\eta)+\left(\frac{3-\mu}{4}\frac{\sin\xi\beta}{\beta}-\frac{1+\mu}{4}\xi\cos\xi\beta\right)\times$$
$$\left.\times\sigma_0(\eta)-\frac{1+\mu}{4}\xi\sin\xi\beta\tau_0(\eta)-(1+\mu)\frac{\partial^2\varphi_0}{\partial\xi\partial\eta}\right\}, \quad (4.84)$$

or, by taking into account (1.238) and (1.246),

$$u(\xi,\eta)=\frac{a}{2}\Big\{U_0(\eta+i\xi)+U_0(\eta-i\xi)-i\frac{1+\mu}{2}\xi U_0'(\eta+i\xi)+$$
$$+i\frac{1+\mu}{2}\xi U_0'(\eta-i\xi)-i\frac{1-\mu}{2}V_0(\eta+i\xi)+i\frac{1-\mu}{2}V_0(\eta-i\xi)-$$
$$-\frac{1+\mu}{2}\xi V_0'(\eta+i\xi)-\frac{1+\mu}{2}\xi V_0'(\eta-i\xi)+$$
$$+\frac{3-\mu}{4i}\int\limits_{\eta-i\xi}^{\eta+i\xi}\sigma_0(\zeta)\,d\zeta-\frac{1+\mu}{4}\xi\sigma_0(\eta+i\xi)-\frac{1+\mu}{4}\xi\sigma_0(\eta-i\xi)+$$
$$+i\frac{1+\mu}{4}\xi\tau_0(\eta+i\xi)-i\frac{1+\mu}{4}\xi\tau_0(\eta-i\xi)-$$
$$-2(1+\mu)\frac{\partial^2\varphi_q}{\partial\xi\partial\eta}\Big\}. \qquad (4.85)$$

If we in the obtained solution everywhere replace number μ by

$$\mu'=\frac{\mu}{1-\mu}, \qquad (4.86)$$

that we will obtain the general solution for a plane strain.

Let us note still that equality (see (4.83))

$$L_{uu}=L_{\sigma\sigma},\quad L_{uv}=L_{\tau\sigma},\quad L_{u\tau}=L_{v\sigma},$$
$$L_{vv}=L_{\tau\tau},\quad L_{\sigma v}=L_{\tau u},\quad L_{vu}=L_{\sigma\tau} \qquad (4.87)$$

is expressed the theorem about the reciprocity of works.

If plate (Fig. 9) is subjected to certain thermal effect, then in two last/latter equations (4.59) will appear supplementary terms and in each of equations (4-62) it will be supplemented on two members, connected with temperature and gradient of temperature on the initial line.

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The general solution of two-dimensional thermoelastic problem is obtained by the literal repetition of presented above. Results are given in the article [42].

§19. Curvature of rectangular plates.

Fine/thin rectangular plate (Fig. 11a) is subjected to the action of transverse load p (kg/cm²). For sagging/deflections w, angles of rotation Θ_x and Θ_y the standards to median surface, the linear bending moments M_x and M_y the given (according to

Kirchhoff), transverse forces V_x and V_y and the concentrated reactions R in angles are accepted as the positive such directions, which are shown in Fig. 11b, and c (vectors of torque/moments right-handed).

By the introduction of dimensionless coordinates (4.57) occupied by plate domain is converted into the domain, analogous given in Fig. 10.

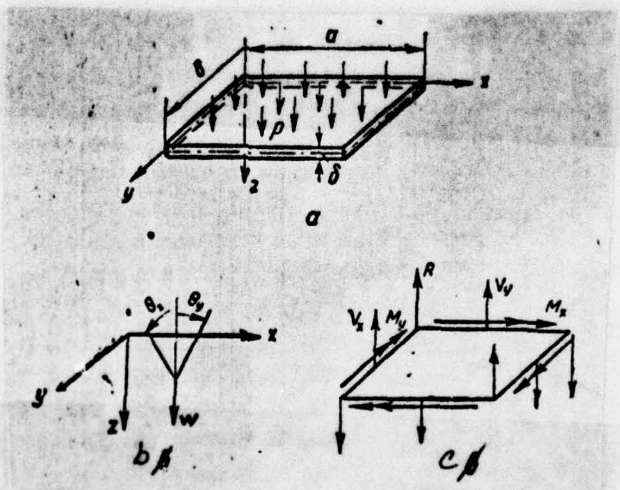

Fig. 11.

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The stressed and the state of strain of the uniform and

isotropic plate of constant thickness δ is described by the complete system of the differential equations

$$\frac{\partial^4 w}{\partial \xi^4} + 2\frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 w}{\partial \eta^4} = \frac{pa^4}{D}; \tag{4.88}$$

$$\left.\begin{aligned}
\theta_x &= -\frac{1}{a}\frac{\partial w}{\partial \eta}, \quad \theta_y = -\frac{1}{a}\frac{\partial w}{\partial \xi},\\
M_x &= -\frac{D}{a^2}\left(\frac{\partial^2 w}{\partial \eta^2} + \mu\frac{\partial^2 w}{\partial \xi^2}\right),\\
M_y &= -\frac{D}{a^2}\left(\frac{\partial^2 w}{\partial \xi^2} + \mu\frac{\partial^2 w}{\partial \eta^2}\right),\\
V_x &= -\frac{D}{a^3}\left[\frac{\partial^3 w}{\partial \xi^3} + (2-\mu)\frac{\partial^3 w}{\partial \xi \partial \eta^2}\right],\\
V_y &= -\frac{D}{a^3}\left[\frac{\partial^3 w}{\partial \eta^3} + (2-\mu)\frac{\partial^3 w}{\partial \xi^2 \partial \eta}\right],\\
R = 2M_{xy} &= -\frac{D}{a^2}(1-\mu)\frac{\partial^2 w}{\partial \xi \partial \eta},
\end{aligned}\right\} \tag{4.89}$$

where, as usual, the flexural rigidity

$$D = \frac{E\delta^3}{12(1-\mu^2)}. \tag{4.90}$$

Let us take for the initial line the edge of plate $\xi = 0$. In this face in accordance with the technical theory of plates have a physical sense and there can be the given values

$$w, \quad \theta = \theta_y, \quad M = M_y, \quad V = V_x. \qquad (4.91)$$

Therefore dimensionless initial functions they will be

$$\left.\begin{aligned} W_0 &= W_0(\eta) = \frac{1}{a} w(0, \eta), \\ \theta_0 &= \theta_0(\eta) = \theta(0, \eta) = \theta_y(0, \eta), \\ M_0 &= M_0(\eta) = \frac{a}{D} M(0, \eta) = \frac{a}{D} M_y(0, \eta), \\ V_0 &= V_0(\eta) = \frac{a^2}{D} V(0, \eta) = \frac{a^2}{D} V_x(0, \eta), \end{aligned}\right\} \qquad (4.92)$$

a the general solution in canonical form (3.37) it will be written as follows:

$$\left.\begin{aligned}
w(\xi,\eta) &= a\{L_{ww}W_0 + L_{w\theta}\theta_0 + L_{wM}M_0 + L_{wV}V_0\} + w_p,\\
\theta_x(\xi,\eta) &= L_{\theta_x w}W_0 + L_{\theta_x\theta}\theta_0 + L_{\theta_x M}M_0 + L_{\theta_x V}V_0 + \theta_{xp},\\
\theta_y(\xi,\eta) &= L_{\theta w}W_0 + L_{\theta\theta}\theta_0 + L_{\theta M}M_0 + L_{\theta V}V_0 + \theta_p,
\end{aligned}\right\}$$

$$\left.\begin{aligned}
M_x(\xi,\eta) &= \frac{D}{a}\{L_{M_x w}W_0 + L_{M_x\theta}\theta_0 + \\
&\quad + L_{M_x M}M_0 + L_{M_x V}V_0\} + M_{xp},\\
M_y(\xi,\eta) &= \frac{D}{a}\{L_{Mw}W_0 + L_{M\theta}\theta_0 + \\
&\quad + L_{MM}M_0 + L_{MV}V_0\} + M_p,\\
V_x(\xi,\eta) &= \frac{D}{a^2}\{L_{Vw}W_0 + L_{V\theta}\theta_0 + \\
&\quad + L_{VM}M_0 + L_{VV}V_0\} + V_p,\\
V_y(\xi,\eta) &= \frac{D}{a^2}\{L_{V_y w}W_0 + L_{V_y\theta}\theta_0 + \\
&\quad + L_{V_y M}M_0 + L_{V_y V}V_0\} + V_{yp},\\
R(\xi,\eta) &= \frac{D}{a}(1-\mu)\{L_{Rw}W_0 + L_{R\theta}\theta_0 + \\
&\quad + L_{RM}M_0 + L_{RV}V_0\} + R_p,
\end{aligned}\right\} \quad (4.93)$$

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Here w_p is a particular solution to nonhomogeneous equation (4.88), that satisfies the conditions

$$\left.\frac{\partial^{j+k} w_p}{\partial \xi^j \partial \eta^k}\right|_{\xi=0} = 0, \text{ if } 0 < j+k < 3. \qquad (4.94)$$

a $\theta_{xp}, \theta_p, \ldots R_p$ - the functions, which will be obtained, if we in formulas (4.89) w replace by w_p. According to (4.10) - (4.12) it is possible to take

$$w_p = -\frac{a^4}{4D}\int\limits_0^\xi d\zeta \int\limits_{\eta-i(\xi-\zeta)}^{\eta+i(\xi-\zeta)} dt \int\limits_0^\zeta d\zeta' \int\limits_{t-i(\zeta-\zeta')}^{t+i(\zeta-\zeta')} p(\zeta', t')\,dt'. \qquad (4.95)$$

Specifically, for the evenly distributed load let us have

$$w_p = \frac{pa^4}{24D}\xi^4, \quad \theta_p = \frac{pa^3}{6D}\xi^3, \quad M_{xp} = -\mu\frac{pa^2}{2D}\xi^2,$$
$$M_p = -\frac{pa^2}{2D}\xi^2, \quad V_p = -pa\xi, \quad \theta_{xp} = V_{yp} = R_p = 0, \qquad (4.96)$$

that it corresponds to the cylindrical curvature of the evenly loaded cantilever plate, shown in Fig. 12. In accordance with (4.13) always it will be: with $\xi = 0$

$$w_p = \theta_{xp} = \theta_p = M_{xp} = M_p = V_p = V_{yp} = R_p = 0. \qquad (4.97)$$

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By transfer/converting to the uniform part of the general solution, let us introduce (4.93) in (4.88) and (4.89). As a result we will obtain the following system of differential equations for operator-functions L_{wj}:

$$\left.\begin{aligned}
&\left(\frac{d^4}{d\xi^4}+2\beta^2\frac{d^2}{d\xi^2}+\beta^4\right)L_{wj}=0,\\
&L_{\theta_x j}=\beta L_{wj},\quad L_{\theta j}=\frac{dL_{wj}}{d\xi},\\
&L_{M_x j}=-\left(\mu\frac{d^2}{d\xi^2}+\beta^2\right)L_{wj},\\
&L_{Mj}=-\left(\frac{d^2}{d\xi^2}+\mu\beta^2\right)L_{wj},\\
&L_{Vj}=-\left[\frac{d^3}{d\xi^3}+(2-\mu)\beta^2\frac{d}{d\xi}\right]L_{wj},\\
&L_{V_x j}=-\left[\beta^3+(2-\mu)\beta\frac{d^2}{d\xi^2}\right]L_{wj},\\
&L_{Rj}=-\beta\frac{dL_{wj}}{d\xi},\quad j=w,\ \theta,\ M,\ V.
\end{aligned}\right\}\qquad(4.98)$$

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After integrating this system under the conditions (see (3.34))

$$L_{sj}(0, \beta) = \delta_{sj}; \quad s, j = w, \theta, M, V. \tag{4.99}$$

let us find (again let us lower intermediate lining/calculations)

$$\left.\begin{aligned}
L_{ww} = L_{VV} &= \cos \xi\beta + \frac{1-\mu}{2} \xi\beta \sin \xi\beta, \\
L_{w\theta} = L_{MV} &= \frac{1+\mu}{2} \frac{\sin \xi\beta}{\beta} + \frac{1-\mu}{2} \xi \cos \xi\beta, \\
L_{wM} = L_{\theta V} &= -\frac{1}{2} \xi \frac{\sin \xi\beta}{\beta}, \\
L_{\theta w} = L_{VM} &= -\frac{1+\mu}{2} \beta \sin \xi\beta + \frac{1-\mu}{2} \xi\beta^2 \cos \xi\beta, \\
L_{\theta\theta} = L_{MM} &= \cos \xi\beta - \frac{1-\mu}{2} \xi\beta \sin \xi\beta, \\
L_{Mw} = L_{V\theta} &= \frac{(1-\mu)^2}{2} \xi\beta^2 \sin \xi\beta;
\end{aligned}\right\} \tag{4.100}$$

$$
\left.\begin{aligned}
L_{wV} &= -\frac{1}{2}\frac{\sin\xi\beta}{\beta^2} + \frac{1}{2}\xi\frac{\cos\xi\beta}{\beta^2},\\
L_{\theta_x w} &= \beta\cos\xi\beta + \frac{1-\mu}{2}\xi\beta^2\xi\sin\xi\beta,\\
L_{\theta_x\theta} &= \frac{1+\mu}{2}\sin\xi\beta - \frac{1-\mu}{2}\xi\beta\cos\xi\beta,\\
L_{\theta_x M} &= -\frac{1}{2}\xi\sin\xi\beta,\\
L_{\theta M} &= -\frac{1}{2}\frac{\sin\xi\beta}{\beta} - \frac{1}{2}\xi\cos\xi\beta,\\
L_{M_x w} &= -(1-\mu^2)\beta^2\cos\xi\beta - \frac{(1-\mu)^2}{2}\xi\beta^3\sin\xi\beta,\\
L_{M_x\theta} &= -\frac{(1-\mu)^2}{2}\beta\sin\xi\beta - \frac{(1-\mu)^2}{2}\xi\beta^2\cos\xi\beta,\\
L_{M_x M} &= \mu\cos\xi\beta + \frac{1-\mu}{2}\xi\beta\sin\xi\beta,\\
L_{M_x V} &= \frac{1+\mu}{2}\frac{\sin\xi\beta}{\beta} - \frac{1-\mu}{2}\xi\cos\xi\beta,\\
L_{M\theta} &= \frac{(1-\mu)(3+\mu)}{2}\beta\sin\xi\beta + \frac{(1-\mu)^2}{2}\xi\beta^2\cos\xi\beta,\\
L_{Vw} &= \frac{(1-\mu)(3-\mu)}{2}\beta^3\sin\xi\beta - \frac{(1-\mu)^2}{2}\beta^4\xi\cos\xi\beta,\\
L_{V_y w} &= -(1-\mu)^2\beta^3\cos\xi\beta + \frac{(1-\mu)^2}{2}\xi\beta^4\sin\xi\beta,\\
L_{V_y\theta} &= \frac{(5-\mu)(1-\mu)}{2}\beta^2\sin\xi\beta + \frac{(1-\mu)^2}{2}\xi\beta^3\cos\xi\beta,\\
L_{V_y M} &= (2-\mu)\beta\cos\xi\beta - \frac{1-\mu}{2}\xi\beta^2\sin\xi\beta,\\
L_{V_y V} &= \frac{3-\mu}{2}\sin\xi\beta + \frac{1-\mu}{2}\xi\beta\cos\xi\beta,\\
L_{RW} &= \frac{1+\mu}{2}\beta^3\sin\xi\beta - \frac{1-\mu}{2}\xi\beta^3\cos\xi\beta,
\end{aligned}\right\} \quad (4.101)
$$

$$
\left.\begin{aligned}
L_{R\theta} &= -\beta\cos\xi\beta + \frac{1-\mu}{2}\xi\beta^2\sin\xi\beta,\\
L_{RM} &= \frac{1}{2}\sin\xi\beta + \frac{1}{2}\xi\beta\cos\xi\beta,\\
L_{RV} &= \frac{1}{2}\xi\sin\xi\beta.
\end{aligned}\right\} \quad (4.101)
$$

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Equalities (4.100) can be construed as the peculiar expression of Betty's theorem. In fact, everything it is possible to write them in the form

$$L_{sj} = L_{\bar{j}\bar{s}} \quad (j, s = w, \theta, M, V). \tag{4.102}$$

Then $\bar{j}$ will be those by the power factor, which can produce work on displacement/movement j (if j - displacement/movement), or thereby by the displacement/movement, during which j can produce work (if j is a power factor). This is related also to $\bar{s}$ and s.

§20. the curvature of the plates, referred to polar coordinates [1].

FOOTNOTE [1]. During the writing of the present section partially were used the results of the investigations of N. N. Cherny. ENDFOOTNOTE.

For the plates whose outline/contour is outlined by the coordinate lines of polar coordinate system (Fig. 13a), is logical the solution to carry out in polar coordinates.

The positive directions of sagging/deflections w, of the angles of rotation of standard to median surface ϑ_r and ϑ_θ, the given bending moments M_r and M_θ, the given (Kirchhoff) transverse forces V_r and V_θ and the concentrated reactions V_R in angles are given in Fig. 13b and c (vectors M and ϑ right-handed).

Let us introduce the given polar coordinates (see Section 3 §17)

$$\xi = \ln\frac{r}{R}, \quad r = Re^{\xi}, \quad \left(\lambda = \ln\frac{R_1}{R_2}\right). \qquad (4.103)$$

Fig. 12.

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Then the domain, occupied with plate (curvilinear rectangle ABCD in Fig. 13a) is converted into rectangle ABCD in Fig. 14, and the complete system of the differential equations, which describe the stressed and the state of strain of the uniform and isotropic plate of constant thickness δ, it will take the form:

$$\Delta\Delta w = \frac{pR^4}{D}; \tag{4.104}$$

$$\left.\begin{aligned}
&\vartheta_r = \frac{1}{R} e^{-\xi} \frac{\partial w}{\partial \xi}, \quad \vartheta_\theta = \frac{1}{R} e^{-\xi} \frac{\partial w}{\partial \theta},\\
&M_r = -\frac{D}{R^2} e^{-2\xi}\left[\frac{\partial^2 w}{\partial \xi^2} - (1-\mu)\frac{\partial w}{\partial \xi} + \mu \frac{\partial^2 w}{\partial \theta^2}\right],\\
&M_\theta = -\frac{D}{R^2} e^{-2\xi}\left[\mu \frac{\partial^2 w}{\partial \xi^2} + (1-\mu)\frac{\partial w}{\partial \xi} + \frac{\partial^2 w}{\partial \theta^2}\right],\\
&V_r = -\frac{D}{R^3} e^{-3\xi}\left[\frac{\partial^3 w}{\partial \xi^3} - 2\frac{\partial^2 w}{\partial \xi^2} + (2-\mu) \times \right.\\
&\qquad \left. \times \frac{\partial^3 w}{\partial \xi \partial \theta^2} - (3-\mu)\frac{\partial^2 w}{\partial \theta^2}\right],\\
&V_\theta = -\frac{D}{R^3} e^{-3\xi}\left[(2-\mu)\frac{\partial^3 w}{\partial \xi^2 \partial \theta} - 3(1-\mu) \times \right.\\
&\qquad \left. \times \frac{\partial^2 w}{\partial \xi \partial \theta} + \frac{\partial^3 w}{\partial \theta^3} + 2(1-\mu)\frac{\partial w}{\partial \theta}\right],\\
&V_R = 2M_{r\theta} = -\frac{D}{R^2}(1-\mu) e^{-2\xi}\left(\frac{\partial^2 w}{\partial \xi \partial \theta} - \frac{\partial w}{\partial \theta}\right).
\end{aligned}\right\} \tag{4.105}$$

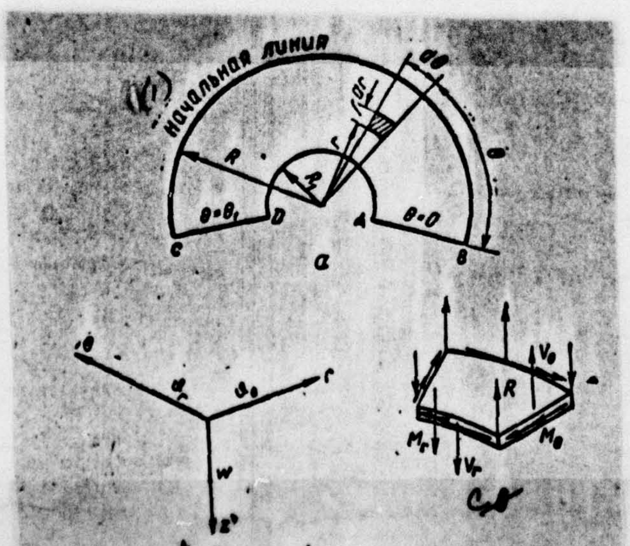

Fig. 13.

Key: (1). the initial line.

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In this case in accordance with (4.39)

$$\Delta\Delta = e^{-2\xi}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\theta^2}\right)\left\{e^{-2\xi}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\theta^2}\right)\right\}. \qquad (4.106)$$

so that instead of (4.104) it will be

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial\xi^2} - 4\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\theta^2} + 4\right)w = e^{4\xi}\frac{pR^4}{D}. \qquad (4.107)$$

Let us take for the initial line the external arc edge of plate ($r = R$, $\xi = 0$). In this face make physical sense and can be assigned/prescribed values

$$\begin{gathered} w(\xi,\theta), \quad \vartheta(\xi,\theta) = \vartheta_r(\xi,\theta), \\ M(\xi,\theta) = M_r(\xi,\theta), \quad V(\xi,\theta) = V_r(\xi,\theta). \end{gathered} \qquad (4.108)$$

Therefore dimensionless initial functions they will be

$$\begin{gathered} W_0 = W_0(\theta) = \frac{1}{R}w(0,\theta), \quad \vartheta_0 = \vartheta_0(\theta) = \vartheta(0,\theta) = \vartheta_r(0,\theta), \\ M_0 = M_0(\theta) = \frac{R}{D}M(0,\theta) = \frac{R}{D}M_r(0,\theta), \\ V_0 = V_0(\theta) = \frac{R^2}{D}V(\xi,\theta) = \frac{R^2}{D^2}V_r(\xi,\theta). \end{gathered} \qquad (4.109)$$

a the general solution in canonical form (3.37) it will be written as follows:

$$\left.\begin{aligned}
w(\xi,\theta) &= R\{L_{ww}W_0 + L_{w\vartheta}\vartheta_0 + L_{wM}M_0 + L_{wV}V_0\} + w_p,\\
\vartheta_r(\xi,\theta) &= L_{\vartheta w}W_0 + L_{\vartheta\vartheta}\vartheta_0 + L_{\vartheta M}M_0 + L_{\vartheta V}V_0 + \vartheta_p,\\
\vartheta_\theta(\xi,\theta) &= L_{\vartheta_\theta w}W_0 + L_{\vartheta_\theta\vartheta}\vartheta_0 + L_{\vartheta_\theta M}M_0 + L_{\vartheta_\theta V}V_0 + \vartheta_{\theta p},\\
M_r(\xi,\theta) &= \frac{D}{R}\{L_{Mw}W_0 + L_{M\vartheta}\vartheta_0 + L_{MM}M_0 + L_{MV}V_0\} + M_p,\\
M_\theta(\xi,\theta) &= \frac{D}{R}\{L_{M_\theta w}W_0 + L_{M_\theta\vartheta}\vartheta_0 +\\
&\quad + L_{M_\theta M}M_0 + L_{M_\theta V}V_0\} + M_{\theta p},\\
V_r(\xi,\theta) &= \frac{D}{R^2}\{L_{Vw}W_0 + L_{V\vartheta}\vartheta_0 + L_{VM}M_0 + L_{VV}V_0\} + V_p,\\
V_\theta(\xi,\theta) &= \frac{D}{R^2}\{L_{V_\theta w}W_0 + L_{V_\theta\vartheta}\vartheta_0 + L_{V_\theta M}M_0 + L_{V_\theta V}V_0\} + V_{\theta p},\\
V_R(\xi,\theta) &= \frac{D}{R}(1-\mu)\{L_{Rw}W_0 + L_{R\vartheta}\vartheta_0 +\\
&\quad + L_{RM}M_0 + L_{RV}V_0\} + V_{Rp}.
\end{aligned}\right\} \quad (4.110)$$

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METHOD OF INITIAL FUNCTIONS FOR TWO DIMENSIONAL BOUNDARY PROBLE--ETC(U)
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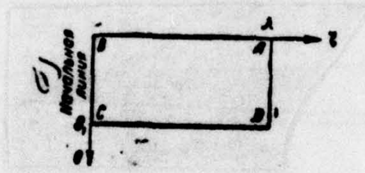

Fig. 14.

Key: (1). the initial line.

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Here w_p is a particular solution to nonhomogeneous equation (4.107), that satisfies the conditions

$$\left.\frac{\partial^{l+k} w_p}{\partial \xi^l \partial \theta^k}\right|_{\xi=0} = 0, \text{ if } 0 < l+k < 2, \qquad (4.111)$$

a $\Phi_p, \Phi_{0_p}, \ldots, \Phi_{k_p}$ - the functions, which will be obtained, if we in formulas (4.105) w replace by w_p.

After introducing the designations

$$\alpha = \frac{\partial}{\partial \xi}, \quad \beta = \frac{\partial}{\partial \theta}, \qquad (4.112)$$

let us rewrite equation (4.107)

$$(\alpha^2+\beta^2)[(\alpha-2)^2+\beta^2]\,w=\frac{R^4}{D}e^{4\xi}p(\xi,\theta). \qquad (4.113)$$

Hence

$$w_p=\frac{R^4}{D}\cdot\frac{1}{\alpha^2+\beta^2}\left\{\frac{1}{(\alpha-2)^2+\beta^2}\{e^{4\xi}p(\xi,\theta)\}\right\}. \qquad (4.114)$$

But on the basis (1.356) and (4.5) will be

$$\frac{1}{(\alpha-2)^2+\beta^2}\{e^{4\xi}p(\xi,\theta)\}=e^{2\xi}\frac{1}{\alpha^2+\beta^2}\{e^{2\zeta}p(\zeta,\theta)\}=$$
$$=-\frac{1}{2i}e^{2\xi}\int\limits_0^{\xi}e^{2\zeta}d\zeta\int\limits_{\theta-i(\xi-\zeta)}^{\theta+i(\xi-\zeta)}p(\zeta,t)\,dt. \qquad (4.115)$$

Introducing this expression in (4.114) and again using (4.5), we obtain

$$w_p=-\frac{R^4}{4D}\int\limits_0^{\xi}e^{2\zeta}d\zeta\int\limits_{\theta-i(\xi-\zeta)}^{\theta+i(\xi-\zeta)}dt\int\limits_0^{\zeta}e^{2\zeta'}d\zeta'\int\limits_{t-i(\zeta-\zeta')}^{t+i(\zeta-\zeta')}p(\zeta',t')\,dt'. \qquad (4.116)$$

By the direct substitution of this expression (4.107) and

(4.105) it is possible to ascertain that w_p satisfies equation (4.107) and, furthermore, with $\xi = 0$

$$w_p = \vartheta_p = \vartheta_{\theta p} = M_p = M_{\theta p} = V_p = V_{\theta p} = V_{Rp} = 0. \quad (4.117)$$

By transfer/converting to the uniform part of the general solution, let us introduce (4.110) in (4.107) and (4.105).

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As a result we will obtain the following system of differential equations for operator-functions L_{wj}:

$$\left.\begin{aligned}
&\left(\frac{d^2}{d\xi^2}+\beta^2\right)\left[\frac{d^2}{d\xi^2}-4\frac{d}{d\xi}+\beta^2+4\right]L_{wj}=0,\\
&L_{\vartheta j}=e^{-\xi}\frac{dL_{wj}}{d\xi},\\
&L_{Mj}=e^{-2\xi}\left[\frac{d^2L_{wj}}{d\xi^2}-(1-\mu)\frac{dL_{wj}}{d\xi}+\mu\beta^2\right]L_{wj},\\
&L_{Vj}=e^{-2\xi}\left[\frac{d^3L_{wj}}{d\xi^3}-2\frac{d^2L_{wj}}{d\xi^2}+(2-\mu)\beta^2\times\right.\\
&\qquad\left.\times\frac{dL_{Mj}}{d\xi}-(3-\mu)\beta^2L_{wj}\right],\\
&L_{\vartheta_\theta j}=e^{-\xi}\beta L_{wj},\\
&L_{M_\theta j}=e^{-2\xi}\left[\mu\frac{d^2L_{wj}}{d\xi^2}+(1-\mu)\frac{dL_{wj}}{d\xi}+\beta^2\right]L_{wj},\\
&L_{V_\theta j}=e^{-2\xi}\left\{(2-\mu)\beta\frac{d^2L_{wj}}{d\xi^2}-3(1-\mu)\beta\times\right.\\
&\qquad\left.\times\frac{dL_{wj}}{d\xi}+\beta[\beta^2+2(1-\mu)]\right\}L_{wj},\\
&L_{V_Rj}=(1-\mu)e^{-2\xi}\beta\left(\frac{dL_{wj}}{d\xi}-L_{wj}\right),\quad j=w,\vartheta,M,V.
\end{aligned}\right\} \quad (4.118)$$

After integrating this system under the conditions

$$L_{sj}(0,\beta)=\delta_{sj};\qquad s,j=w,\vartheta,M,V, \tag{4.119}$$

let us find

$$\left.\begin{aligned}
L_{ww}&=\cos\xi\beta-\frac{1-\mu}{4}\beta\sin\xi\beta+\frac{1-\mu}{4}e^{2\xi}\beta\sin\xi\beta,\\
L_{w\vartheta}&=-\frac{1-\mu}{4}\cos\xi\beta+\frac{1+\mu}{2}\frac{\sin\xi\beta}{\beta}+\frac{1-\mu}{4}e^{2\xi}\cos\xi\beta,\\
L_{wM}&=\frac{1}{4}\Big[-\cos\xi\beta-(2+\beta^2)\frac{\sin\xi\beta}{\beta}+\\
&\qquad+e^{2\xi}(\cos\xi\beta+\beta\sin\xi\beta)\Big]\frac{1}{1+\beta^2},\\
L_{wV}&=\frac{1}{4}\left[\cos\xi\beta+\frac{\sin\xi\beta}{\beta}-e^{2\xi}\left(\cos\xi\beta-\frac{\sin\xi\beta}{\beta}\right)\right]\cdot\frac{1}{1+\beta^2},
\end{aligned}\right\} \tag{4.120}$$

$$\left.\begin{aligned}
L_{\vartheta w}&=e^{-\xi}\Big[-\frac{1-\mu}{4}\beta^2\cos\xi\beta-\beta\sin\xi\beta+\\
&\qquad+\frac{1-\mu}{4}e^{2\xi}\beta(\beta\cos\xi\beta+2\sin\xi\beta)\Big]
\end{aligned}\right\} \tag{4.120}$$

and so forth.

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During the solution to specific problems it can turn out to be more convenient to choose the initial line not on curvilinear, but at the rectilinear (radial) edge of plate ($\theta = 0$). As dimensionless initial functions here will be

$$\left.\begin{aligned} W_0 &= W_0(\xi) = \frac{1}{R} w(\xi, 0), & \Phi_0 &= \Phi_0(\xi) = e^{\xi}\vartheta_{\theta}(\xi, 0), \\ M_0 &= M_0(\xi) = \frac{R}{D} e^{2\xi} M_{\theta}(\xi, 0), & V_0 &= V_0(\xi) = \frac{R^2}{D} e^{3\xi} V_{\theta}(\xi, 0). \end{aligned}\right\} \quad (4.121)$$

a the operators will take the form

$$\left.\begin{aligned} L_{ww} &= \cos\theta\alpha \cdot \left[1 - \frac{(1-\mu)\alpha(\alpha-1)}{2(\alpha-2)}\right] + \\ &\quad + \cos\theta(\alpha-2) \cdot \frac{(1-\mu)\alpha(\alpha-1)}{2(\alpha-2)}, \\ L_{w\Phi} &= \frac{1}{4}[(1-\mu)\alpha + 2(1+\mu)] \frac{\sin\theta\alpha}{\alpha} - \\ &\quad - \frac{1-\mu}{4} \sin\theta(\alpha-2) \end{aligned}\right\} \quad (4.122)$$

and so forth.

The construction of the common/general/total and quotient of solutions, all operators, and also some other formulas are given with to article [3].

§21. Natural oscillations of rectangular plates.

The free oscillation/vibrations of rectangular plate upon the linear formulation of the problem are described by system of equations (4.89), and instead of (4.88) will be

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}-\nu^4 w=0, \qquad (4.123)$$

where

$$\nu=\frac{\gamma\delta\omega^2}{gD}, \qquad (4.124)$$

ω - angular frequency, γ - the specific gravity/weight of the material of plate, g - the acceleration of gravity.

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The positive directions of displacement/movements and static values

will be the same as in Fig. 11.

Accepting as the initial line edge $\xi = 0$, we they must as initial select the same functions (4.92), and the general solution to seek in the form (4.93), after reject/throwing only in it terms w_p, θ_{xp} θ_{xp} and so forth. After substituting those which were altered thus function (4.93) into equations (4.123) and (4.89), we will obtain for determining operator-functions L_{sl} the system of differential equations, which differs from (4.98) only in terms of first equation. Specifically, instead of it it will be

$$\frac{d^4L_{wl}}{d\xi^4}+2\beta^2\frac{d^2L_{wl}}{d\xi^2}+(\beta^4-\nu^4)L_{wl}=0. \tag{4.125}$$

After integrating the obtained system under the conditions (4.99), let us find all operators:

$$\left.\begin{aligned}
L_{ww} &= -\frac{(1-\mu)\beta^2-\nu^2}{2\nu^2}\cos\xi\sqrt{\beta^2+\nu^2}+\\
&\quad+\frac{(1-\mu)\beta^2+\nu^2}{2\nu^2}\cos\xi\sqrt{\beta^2-\nu^2},\\
L_{w\theta} &= \frac{(1-\mu)\beta^2+\nu^2}{2\nu^2}\cdot\frac{\sin\xi\sqrt{\beta^2+\nu^2}}{\sqrt{\beta^2+\nu^2}}+\\
&\quad+\frac{(1-\mu)\beta^2-\nu^2}{2\nu^2}\cdot\frac{\sin\xi\sqrt{\beta^2-\nu^2}}{\sqrt{\beta^2-\nu^2}},\\
L_{wM} &= -\frac{1}{2\nu^2}\cos\xi\sqrt{\beta^2+\nu^2}+\frac{1}{2\nu^2}\cos\xi\sqrt{\beta^2-\nu^2}
\end{aligned}\right\} \tag{4.126}$$

and so forth.

The construction of general solution, all operators, and also the resolving equations for the basic cases of the attachment of plate during symmetrical, skew-symmetric and asymmetric oscillation/vibrations are given in the article [10].

let us note still that in spite of the apparentirrationality,

and, and also, therefore, the singularity of operators (4. 126), actually they all are regular. In this it is easy to verify that after decomposing operator-functions $\cos\xi\sqrt{\beta^2\pm\nu^2}$, $\sin\xi\sqrt{\beta^2\pm\nu^2}$ according to degrees ξ.

Chapter 5.

APPLICATION/USE OF A METHOD OF THE INITIAL FUNCTIONS TO THE SOLUTION TO THE APPLIED PROBLEMS OF MECHANICS.

§22. Free twisting the rod of semicircular section.

The function of twisting Prandtl, which describes free (Saint-Venant) twisting the cylindrical rod, cross section of which is shown in Fig. 15, satisfies the equation

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2}{\partial \beta^2}\right)\times \Phi = -2. \tag{5.1}$$

a voltage/stress it is determined from the formulas

$$\tau_{rz} = \frac{G\theta}{r}\cdot\frac{\partial \Phi}{\partial \beta}, \quad \tau_{\beta z} = -G\theta\frac{\partial \Phi}{\partial r}. \tag{5.2}$$

where G - shear modulus, ϑ - the linear angle of torsion.

On the outline/contour of domain $\Phi = 0$ after passing to new variable (see (4.37))

$$\xi = \ln\frac{r}{R}, \quad r = Re^{\xi}, \quad \lambda = \frac{R_1}{R}, \tag{5.3}$$

we will obtain the equations

$$\frac{\partial^2\Phi}{\partial\xi^2} + \frac{\partial^2\Phi}{\partial\theta^2} = -2R^2e^{2\xi}, \tag{5.4}$$

$$\tau_{rz} = \frac{G\vartheta}{R}e^{-\xi}\frac{\partial\Phi}{\partial\theta}, \quad \tau_{\theta z} = -\frac{G\vartheta}{R}e^{-\xi}\frac{\partial\Phi}{\partial\xi} \tag{5.5}$$

and the boundary conditions:

$$\Phi(0, \theta) = 0, \tag{5.6}$$

$$\Phi(\ln\lambda, \theta) = 0, \tag{5.7}$$

$$\Phi(\xi, \pm\alpha) = 0. \tag{5.8}$$

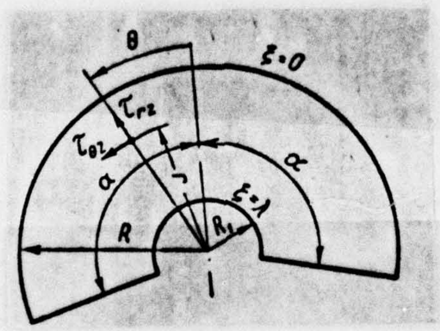

Fig. 15.

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Is consistent initial to line with the initial arc edge (ξ = 0) and will use formulas (4.42) - (4.44)

$$\Phi(\xi, \theta) = \cos \xi\beta \Phi_0(\theta) + \frac{\sin \xi\beta}{\beta} \Phi_1(\theta) + \Phi_p. \qquad (5.9)$$

where, according to (5.6),

a

$$\Phi_0 = \Phi(0, \theta) = 0. \qquad (5.10)$$

$$\Phi_1(\theta) = \left.\frac{\partial \Phi}{\partial \xi}\right|_{\xi=0} \qquad (5.11)$$

is an unknown function. Particular solution we find through (4.43)

$$\Phi_1 = -\frac{2R^2}{2i}\int_0^{\xi} dt \int_{\theta - i(\xi - t)}^{\theta + i(\xi - t)} e^{2\zeta} d\zeta = R^2\left(\xi + \frac{1}{2} - \frac{1}{2}e^{2\xi}\right). \quad (5.12)$$

Thus,

$$\Phi(\xi, \theta) = \frac{\sin \xi\beta}{\beta}\Phi_1(\theta) + R^2\left(\xi + \frac{1}{2} - \frac{1}{2}e^{2\xi}\right). \quad (5.13)$$

We satisfy condition (5.17)

$$\frac{\sin(\ln\lambda\cdot\beta)}{\beta}\Phi_1(\theta) = R^2\left(\frac{\lambda^2 - 1}{2} - \ln\lambda\right). \quad (5.14)$$

After expanding operator in a series in β

$$\ln\lambda - \frac{(\ln\lambda)^3}{3!}\beta^2 + \ldots, \quad (5.15)$$

easily we find the particular solution

$$\Phi_1(\theta)_{\text{частн}} = R^2\left(\frac{\lambda^2 - 1}{2\ln\lambda} - 1\right). \quad (5.16)$$

To homogeneous equation (5.14) corresponds the characteristic equation

$$\frac{\sin(k\ln\lambda)}{k} = 0. \quad (5.17)$$

roots of which will be the number

$$k_n = \pm \frac{n\pi}{\ln \lambda}, \quad n = 1, 2, 3, \ldots \tag{5.18}$$

Consequently,

$$\Phi_1(\theta) = R^2\left(\frac{\lambda^2 - 1}{2 \ln \lambda} - 1\right) + \sum_{n=1}^{\infty}\left(A_n \operatorname{ch} \frac{n\pi\theta}{\ln \lambda} + B_n \operatorname{sh} \frac{n\pi\theta}{\ln \lambda}\right). \tag{5.19}$$

By introducing (5.19) in (5.14) and by taking into account formula (1.246) and (1.205), let us find

$$\Phi(\xi, \theta) = \frac{R^2}{2}\left(\frac{\lambda^2 - 1}{\ln \lambda}\xi + 1 - e^{2\xi}\right) + \\ + \sum_{n=1}^{\infty}\left(A_n \operatorname{ch} \frac{n\pi\theta}{\ln \lambda} + B_n \operatorname{sh} \frac{n\pi\theta}{\ln \lambda}\right)\frac{\ln \lambda}{n\pi} \sin \frac{n\pi\xi}{\ln \lambda}. \tag{5.20}$$

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By satisfying now conditions (5.8), let us arrive at the equalities

$$\sum_{n=1}^{\infty} \frac{1}{n} A_n \operatorname{ch} \frac{n\pi\alpha}{\ln \lambda} \sin \frac{n\pi\xi}{\ln \lambda} = \frac{\pi R^2}{4 \ln \lambda}\left(\frac{1 - \lambda^2}{\ln \lambda}\xi + e^{2\xi} - 1\right), \tag{5.21}$$

$$B_n = 0. \tag{5.22}$$

In order to find coefficients A_n, it suffices to multiply (5.21) by sine $n\pi\xi/\ln\lambda \, d\xi$ and to integrate over ξ within limits from $\xi = 0$ to $\xi = \ln\lambda$. As a result we will obtain

$$\Phi(\xi, \vartheta) = \frac{R^2}{2}\left\{1 - \frac{1-\lambda^2}{\ln\lambda}\xi - e^{2\xi} - \right.$$

$$\left. - \frac{8(\ln\lambda)^2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{1-\lambda^2\cdot(-1)^n}{(2\ln\lambda)^2+(n\pi)^2}\cdot\frac{\operatorname{ch}\frac{n\pi\vartheta}{\ln\lambda}}{\operatorname{ch}\frac{n\pi\alpha}{\ln\lambda}}\sin\frac{n\pi\xi}{\ln\lambda}\right\}. \quad (5.23)$$

Knowing the function of Prandtl, it is not difficult to fulfill different concrete/specific/actual calculations. Let us plot, for example, the diagram/curve of distribution τ along the axis of the

symmetry of semicircular section ($\alpha = \pi/2$) during $\lambda = 0.5$. According to (5.23) and (5.5) with $\theta = 0$

$$\tau_{\rho z}^{t} = \theta G R \cdot \tilde{\tau}, \tag{5.24}$$

where is marked

$$\tilde{\tau} = e^{-\xi}\left\{\frac{1-\lambda^2}{\ln\lambda} + 2e^{2\xi} + 8\ln\lambda\sum_{n=1}^{\infty}\frac{1-\lambda^2(-1)^n}{(2\ln\lambda)^2+(n\pi)^2}\cdot\frac{\cos\frac{n\pi\xi}{\ln\lambda}}{\mathrm{ch}\,\frac{n\pi^2}{2\ln\lambda}}\right\}. \tag{5..25}$$

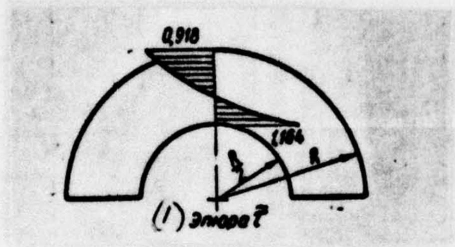

Fig. 16.

Key: (1). Diagram/curve.

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Let us substitute here value λ = 0.5 and return to alternating/variable r

$$\tilde{\tau}(r) = \frac{R}{r}\Big[-1{,}08203 + 2\left(\frac{r}{R}\right)^2 - 0{,}000095132\cos\left(4{,}53240\ln\frac{r}{R}\right) - \ldots\Big]. \quad (5.26)$$

Is here written out the only first term of a series, but even he is

sufficiently small in order that him it would be possible to reject/throw and to count

$$\tilde{\tau}(r) = 2\frac{r}{R} - 1.08203\frac{R}{r}. \tag{5.27}$$

Correspond to this formula of diagram/curve $\tilde{\tau}$ is represented in Fig. 16.

§23. Some resolving equations.

The examined in the preceding/previous paragraph problem is sufficiently simple. In the more complex cases, when it is necessary to implement all those stages of the calculation, about which went the speech into §12 and 13, considerable labor input requires the composition of the resolving equation (see page 132) and, especially, the determination of the roots of characteristic equation (see §14). In the present paragraph are given the resolving equations for the most characteristic conditions of attachment during curvature and the oscillation/vibrations of rectangular plate, and also in two-dimensional problem.

The system of transcendental differential equations, obtained as a result of satisfaction to boundary conditions at the edge, parallel

to the initial line, will be, generally speaking, heterogeneous (see (3.44)). However, assuming that the particular solution is found (about this goes speech on page 132), we here write out the only uniform part. The attachments of two parallel edges of plate are shown in Fig. 17-22 (for example, in Fig. 21 left edges hinged is supported, right - is free). The initial line in all cases coincides with axle/axis η. If strain is symmetrical (or is skew-symmetric) relative to the axis of the symmetry of plate, then axle/axis η expedient to choose in the manner that this is done in Fig. 17-19.

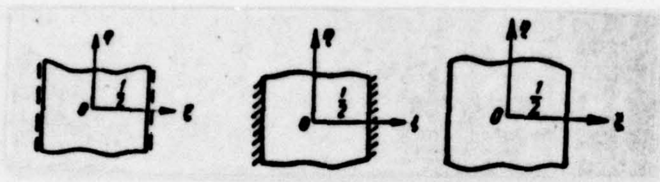

Fig. 17. Fig. 18. Fig. 19.

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1. Curvature and the oscillation/vibrations of plate.

Symmetrical strain for Fig. 17

$$\left.\begin{aligned}&W_0 = L_{wM}\left(\frac{1}{2},\ \beta\right)\varphi(\eta), \quad M_0 = -L_{ww}\left(\frac{1}{2},\ \beta\right)\varphi(\eta),\\ &\cos^2\frac{\beta}{2}\varphi(\eta) = 0, \quad \cos\frac{\sqrt{\beta^2+\nu^2}}{2}\cos\frac{\sqrt{\beta^2-\nu^2}}{2}\varphi(\eta) = 0;\end{aligned}\right\} \quad (5.28)$$

for Fig. 18

$$\left.\begin{aligned}&W_0 = L_{wM}\left(\frac{1}{2},\ \beta\right)\varphi(\eta), \quad M_0 = -L_{ww}\left(\frac{1}{2},\ \beta\right)\varphi(\eta),\\ &\left(1+\frac{\sin\beta}{\beta}\right)\psi(\eta) = 0,\\ &\left(\sqrt{\beta^2+\nu^2}\sin\frac{\sqrt{\beta^2+\nu^2}}{2}\cos\frac{\sqrt{\beta^2-\nu^2}}{2} - \right.\\ &\left.-\sqrt{\beta^2-\nu^2}\sin\frac{\sqrt{\beta^2-\nu^2}}{2}\cos\frac{\sqrt{\beta^2+\nu^2}}{2}\right)\varphi(\eta) = 0;\end{aligned}\right\} \quad (5.29)$$

for Fig. 19.

$$\left.\begin{aligned}&M_0 = L_{Mw}\left(\frac{1}{2},\ \beta\right)\varphi(\eta), \quad W_0 = -L_{MM}\left(\frac{1}{2},\ \beta\right)\varphi(\eta),\\ &\beta^2[(3+\mu)\sin\beta - (1-\mu)\beta]\varphi(\eta) = 0,\\ &\left\{[(1-\mu)\beta^2+\nu^2]\sqrt{\beta^2-\nu^2}\sin\frac{\sqrt{\beta^2-\nu^2}}{2}\cos\frac{\sqrt{\beta^2+\nu^2}}{2} - \right.\\ &\left.-[(1-\mu)\beta^2-\nu^2]\sqrt{\beta^2+\nu^2}\sin\frac{\sqrt{\beta^2+\nu^2}}{2}\cos\frac{\sqrt{\beta^2-\nu^2}}{2}\right\}\times\\ &\times\varphi(\eta) = 0.\end{aligned}\right\} \quad (5.30)$$

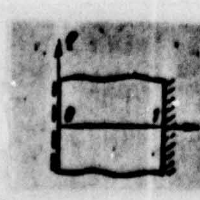

Fig. 20.

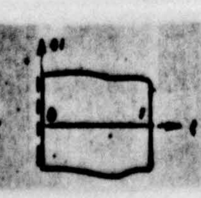

Fig. 21.

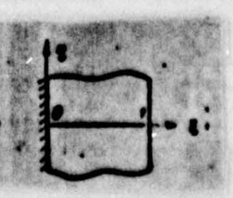

Fig. 22.

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Skew-symmetric strain

for Fig. 17

$$\left.\begin{aligned}
&\theta_0 = L_{wV}\left(\frac{1}{2},\beta\right)\varphi(\eta), \qquad V_0 = -L_{w_0}\left(\frac{1}{2},\beta\right)\varphi(\eta),\\
&\frac{1}{\beta^2}\sin^2\frac{\beta}{2}\,\varphi(\eta) = 0,\\
&\frac{1}{\sqrt{\beta^4-\nu^4}}\sin\frac{\sqrt{\beta^2+\nu^2}}{2}\sin\frac{\sqrt{\beta^2-\nu^2}}{2}\,\varphi(\eta) = 0;
\end{aligned}\right\} \quad (5.31)$$

for Fig. 18

$$\left.\begin{aligned}
&\theta_0 = L_{wV}\left(\frac{1}{2},\beta\right)\varphi(\eta), \qquad V_0 = -L_{w_0}\left(\frac{1}{2},\beta\right)\varphi(\eta),\\
&\frac{1}{\beta^2}\left(1-\frac{\sin\beta}{\beta}\right)\varphi(\eta) = 0,\\
&\frac{1}{\sqrt{\beta^2-\nu^2}}\sin\frac{\sqrt{\beta^2-\nu^2}}{2}\cos\frac{\sqrt{\beta^2+\nu^2}}{2} -\\
&-\frac{1}{\sqrt{\beta^2+\nu^2}}\sin\frac{\sqrt{\beta^2+\nu^2}}{2}\cos\frac{\sqrt{\beta^2-\nu^2}}{2}\Bigg)\varphi(\eta) = 0;
\end{aligned}\right\} \quad (5.32)$$

for Fig. 19

$$\left.\begin{array}{c}\theta_0 = L_{MV}\left(\frac{1}{2}, \beta\right)\varphi(\eta), \qquad V_0 = -L_{M\theta}\left(\frac{1}{2}, \beta\right)\varphi(\eta), \\ \beta\left[(1-\mu)\beta + (3+\mu)\sin\beta\right]\varphi(\eta) = 0, \\ \left\{\frac{[(1-\mu)\beta^2 - \nu^2]^2}{\sqrt{\beta^2-\nu^2}}\sin\frac{\sqrt{\beta^2-\nu^2}}{2}\cos\frac{\sqrt{\beta^2+\nu^2}}{2} - \right. \\ \left. -\frac{[(1-\mu)\beta^2 + \nu^2]^2}{\sqrt{\beta^2+\nu^2}}\sin\frac{\sqrt{\beta^2+\nu^2}}{2}\cos\frac{\sqrt{\beta^2-\nu^2}}{2}\right\}\varphi(\eta) = 0;\end{array}\right\} \quad (5.33)$$

Asymmetric strain

for Fig. 20

$$\left.\begin{array}{c}\theta_0 = L_{wV}(1, \beta)\varphi(\eta), \qquad V_0 = L_{w\theta}(1, \beta)\varphi(\eta), \\ \left(1 - \frac{\sin 2\beta}{2\beta}\right)\frac{1}{\beta^2}\varphi(\eta) = 0, \\ \left(\frac{\sin\sqrt{\beta^2-\nu^2}}{\sqrt{\beta^2-\nu^2}}\cos\sqrt{\beta^2+\nu^2} - \right. \\ \left. -\frac{\sin\sqrt{\beta^2+\nu^2}}{\sqrt{\beta^2+\nu^2}}\cos\sqrt{\beta^2-\nu^2}\right)\varphi(\eta) = 0;\end{array}\right\} \quad (5.34)$$

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for Fig. 21

$$\left.\begin{array}{c}
\theta_0 = L_{MV}(1, \beta)\,\varphi(\eta), \qquad V_0 = -L_{M\theta}(1, \beta)\,\varphi(\eta), \\
\beta\left[(1-\mu)\cdot 2\beta + (3+\mu)\sin 2\beta\right]\varphi(\eta) = 0, \\
\left\{\left[(1-\mu)\beta^2 - \nu^2\right]^2 \dfrac{\sin\sqrt{\beta^2-\nu^2}}{\sqrt{\beta^2-\nu^2}}\cos\sqrt{\beta^2+\nu^2} - \right. \\
\left. -\left[(1-\mu)\beta^2 + \nu^2\right]^2 \dfrac{\sin\sqrt{\beta^2+\nu^2}}{\sqrt{\beta^2+\nu^2}} \cos\sqrt{\beta^2-\nu^2}\right\}\varphi(\eta) = 0;
\end{array}\right\} \quad (5.35)$$

for Fig. 22

$$\left.\begin{array}{c} M_0 = L_{MV}(1,\beta)\varphi(\eta), \quad V_0 = -L_{NM}(1,\beta)\varphi(\eta). \\ \left[(1-\mu)^2\beta^2 - \frac{3-2\mu-\mu^2}{2}\cos 2\beta - \frac{5+2\mu+\mu^2}{2}\right]\varphi(\eta) = 0. \\ \left\{2[(1-\mu)^2\beta^4 - \nu^4] - [(1-\mu)\beta^2 - \nu^2]^2\left(\cos\sqrt{\beta^2-\nu^2}\cos\sqrt{\beta^2+\nu^2} + \right.\right. \\ \left. + \sqrt{\frac{\beta^2+\nu^2}{\beta^2-\nu^2}}\sin\sqrt{\beta^2+\nu^2}\sin\sqrt{\beta^2-\nu^2}\right) - \\ - [(1-\mu)\beta^2+\nu^2]^2\left(\cos\sqrt{\beta^2-\nu^2}\cos\sqrt{\beta^2+\nu^2} + \right. \\ \left.\left. + \sqrt{\frac{\beta^2-\nu^2}{\beta^2+\nu^2}}\sin\sqrt{\beta^2+\nu^2}\sin\sqrt{\beta^2-\nu^2}\right)\right\}\varphi(\eta) = 0. \end{array}\right\} \quad (5.36)$$

Here the everywhere that resolving equation, which does not contain parameter ν, is related to the curvature of plate, but containing ν - to oscillation/vibrations.

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2. State of plane stress.

(Necessary designations are given in Fig. 9 and 10)

$$\left.\begin{aligned}
&\text{а)}\quad \sigma_0=\tau_0=0, \qquad \sigma_x(1,\eta)=\tau_x(1,\eta)=0,\\
&U_0=\beta(\sin\beta+\beta\cos\beta)\,\varphi(\eta), \qquad V_0=\beta^2\sin\beta\varphi(\eta),\\
&\beta^2(\beta^2-\sin^2\beta)\,\varphi(\eta)=0;
\end{aligned}\right\}\quad (5.37)$$

$$\left.\begin{aligned}
&\text{б)}\quad U_0=V_0=0, \qquad u(1,\eta)=v(1,\eta)=0,\\
&\sigma_0=(1+\mu)\sin\beta\varphi(\eta), \quad \tau_0\left[(3-\mu)\frac{\sin\beta}{\beta}-(1+\mu)\cos\beta\right]\varphi(\eta),\\
&\left[\frac{\sin^2\beta}{\beta^2}-\left(\frac{1+\mu}{3-\mu}\right)^2\right]\varphi(\eta)=0;
\end{aligned}\right\}\quad (5.38)$$

$$\left.\begin{aligned}
&\text{в)}\quad \sigma_0=\tau_0=0, \qquad u(1,\eta)=v(1,\eta)=0,\\
&U_0=\left(\cos\beta-\frac{1+\mu}{2}\beta\sin\beta\right)\varphi(\eta), \qquad V_0=\left(\frac{1-\mu}{2}\sin\beta+\right.\\
&\left.+\frac{1+\mu}{2}\beta\cos\beta\right)\varphi(\eta),\\
&\left[(3-\mu)(1+\mu)\cos 2\beta-\frac{(1+\mu)^2}{2}(2\beta)^2+4+(1-\mu)^2\right]\varphi(\eta)=0;
\end{aligned}\right\}\quad (5.39)$$

$$\left.\begin{aligned}
&\text{г)}\quad U_0=\tau_0=0, \qquad u(1,\eta)=\tau(1,\eta)=0,\\
&V_0=[(1-\mu)\sin\beta-(1+\mu)\beta\cos\beta]\,\varphi(\eta),\\
&\sigma_0=2(1+\mu)\beta(\sin\beta+\beta\cos\beta)\,\varphi(\eta),\\
&\sin^2\beta\varphi(\eta)=0;
\end{aligned}\right\}\quad (5.40)$$

$$\left.\begin{aligned}
&\text{д)}\quad V_0=\sigma_0=0, \qquad v(1,\eta)=\sigma(1,\eta)=0,\\
&U_0=\left|(3-\mu)\frac{\sin\beta}{\beta}+(1+\mu)\cos\beta\right|\varphi(\eta),\\
&\tau_0=2[(1-\mu)\sin\beta+(1+\mu)\beta\cos\beta]\,\varphi(\eta),\\
&\sin^2\beta\varphi(\eta)=0;
\end{aligned}\right\}\quad (5.41)$$

$$\left.\begin{aligned}
&\text{е)}\quad U_0=\tau_0=0, \qquad \sigma(1,\eta)=\tau(1,\eta)=0,\\
&V_0=\left(\cos\beta+\frac{1+\mu}{2}\beta\sin\beta\right)\varphi(\eta),\\
&\sigma_0=-(1+\mu)\beta^2\sin\beta\varphi(\eta),\\
&\beta\left(\beta+\frac{\sin 2\beta}{2}\right)\varphi(\eta)=0.
\end{aligned}\right\}\quad (5.42)$$

In accordance with the facts that it was said on page 131, the resolving equations allow/assume different interpretation. For example, by taking into account (4.100) and (4.101), instead of (5.32) it is possible tc write

$$\left.\begin{aligned} \theta_0 &= \frac{1}{2}\left(-\frac{\sin\frac{\beta}{2}}{\beta^3} + \frac{1}{2}\,\frac{\cos\frac{\beta}{2}}{\beta^2}\right)\varphi(\eta), \\ V_0 &= -\left(\frac{1+\mu}{2}\cdot\frac{\sin\frac{\beta}{2}}{\beta} + \frac{1-\mu}{4}\cos\frac{\beta}{2}\right)\varphi(\eta); \end{aligned}\right\} \quad (5.43)$$

$$\left(1 - \frac{\sin\beta}{\beta}\right)\cdot\frac{1}{\beta^2}\varphi(\eta) = 0. \quad (5.44)$$

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The last/latter equation - this transcendental ordinary differential equation for the resolving function $\varphi(\eta)$. and formulas (5.43) express the initial functions through resolving, in addition with the help of transcendental differential operations. If we expand in series those entering in (5.43) and (5.44) operators, then we will cbtain:

$$\left.\begin{aligned} \theta_0 &= -\frac{1}{2^3 \cdot 3!}\left[\varphi(\eta) - \frac{1}{40}\varphi''(\eta) + \ldots\right], \\ V_0 &= -\frac{1}{2}\left[\varphi(\eta) - \frac{2-\mu}{2^3 \cdot 3!}\varphi''(\eta) + \frac{3-2\mu}{2^4 \cdot 5!}\varphi^{IV}(\eta) - \ldots\right]; \end{aligned}\right\} \quad (5.45)$$

$$\frac{1}{3!}\varphi(\eta) - \frac{1}{5!}\varphi''(\eta) + \frac{1}{7!}\varphi^{IV}(\eta) - \ldots = 0. \quad (5.46)$$

After being bounded in series to certain number of terms, we will obtain usual differential expressions.

But if we use formulas for the realization of the operators, then resolving equation (5.44) will take the form

$$\int_0^{\eta}(\eta-\zeta)\varphi(\zeta)\,d\zeta - \frac{1}{2l}\int_{\eta-l}^{\eta+l}dt\int_0^{t}(t-\zeta)\varphi(\zeta)\,d\zeta = 0. \quad (5.47)$$

This already there will be integral equation for the resolving function.

§24. Roots of some characteristic equations.

In the preceding/previous paragraph are given the resolving equations for the characteristic forms of the boundary conditions of two-dimensional problem and curvature of plate. If we seek the resolving function in the form $\varphi = e^{\alpha\eta}$, then each resolving equation will reduce to the appropriate characteristic equation (see page 132). For the majority of these equations was found certain quantity of first (counting in the ascending order in the module/modulus) roots. Roots were calculated in the manner that this is described in §14. As an example of calculations Table 1-5 gives iterative process for equation (3.55) with

$$\nu = \frac{1-\mu}{3+\mu} = 0{,}21212, \qquad (5.48)$$

based on formulas (3.68) - (3.72).

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Table 1.

	$s=1,\ \frac{\pi}{2}(4s+1)=7{,}85398$			
1	δ_1	0,10	0,17630	0,17624
2	$x_1=7{,}85398-\delta_1$	7,75398	7,67768	7,67774
3	$\cos\delta_1=\sin x_1$	0,99500	0,98450	0,98451
4	$0{,}21212/(3)^{*}$	0,21319	0,21546	0,21546
5	$\mathrm{ch}\, y_1=(4)\cdot(2)$	1,65307	1,65423	1,65425
6	$y_1=\mathrm{arch}(5)$	1,08836	1,08925	1,08927
7	$\mathrm{sh}\, y_1$	1,31631	1,31779	1,31781
8	$(5)^2-(7)^2$	0,99997	0,99991	0,99992
9	$0{,}21212\cdot(6)$	0,23086	0,2[illegible]105	0,203106
10	$\sin\delta_1=\cos x_1=\frac{(9)}{(7)}$	0,17538	0,17533	0,17534
11	$(3)^2+(10)^2$	1,02078	0,99998	1,00000
12	$\delta_1=\arcsin(10)$	0,17630	0,17624	0,17625

in Table 1-5 for convenience in the calculations in numeral in brackets () is designated the number of the row, from which one should take numerical value for the execution of the corresponding operations.

Table 2.

$s = 2;\ \frac{\pi}{2}(4s + 1) = \frac{9\pi}{2} = 14,13717$				
1	δ_2	0,10	0,13275	0,13263
2	$x_2 = 14,13717 - \delta_2$	14,03717	14,00442	14,00454
3	$\cos\delta_2 = \sin x_2$	0,99500	0,99105	0,99122
4	0,21212/(3)	0,21319	0,21404	0,21400
5	$\mathrm{ch}\, y_2 = (4)\cdot(2)$	2,99258	0,99751	2,99697
6	$y_2 = \mathrm{arch}\,(5)$	1,76013	1,76186	1,76167
7	$\mathrm{sh}\, y_2$	2,82059	2,82578	2,82521
8	$(5)^2 - (7)^2$	0,99981	1,00003	1,00002
9	0,21212·(6)	0,37336	0,37372	0,37368
10	$\sin\delta_2 = \cos x_2 = \frac{(9)}{(7)}$	0,13237	0,13225	0,13227
11	$(3)^2 + (10)^2$	1,00755	0,99967	1,00001
12	$\delta_2 = \arcsin\,(10)$	0,13275	0,13263	0,13265

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Table 3.

$s=3,\ \frac{\pi}{2}(4s+1)=\frac{13\pi}{2}=20,42032$			
1	δ_3	0,10	0,10818
2	$x_3=20,42035-\delta_3$	20,32035	20,31217
3	$\cos\delta_3=\sin x_3$	0,99500	0,99415
4	$0,21212/(3)$	0,21319	0,21337
5	$\mathrm{ch}\,y_3=(4)\cdot(2)$	4,33210	4,33401
6	$y_3=\mathrm{arch}\,(5)$	2,14560	2,14605
7	$\mathrm{sh}\,y_3$	4,21508	4,21702
8	$(5)^2-(7)^2$	1,00019	1,00038
9	$0,21212\cdot(6)$	0,45512	0,45522
10	$\sin\delta_3=\cos x_3=\frac{(9)}{(7)}$	0,10797	0,10795
11	$(3)^2+(10)^2$	1,00168	0,99999
12	$\delta_3=\arcsin\,(10)$	0,10818	0,10816

TABLE 4.

$s=4;\ \frac{\pi}{2}(4s+1)=\frac{17\pi}{2}=26,70353$				
1	δ_4	0,10	0,09192	0,09214
2	$x_4=26,70353-\delta_4$	26,69353	26,61161	26,61139
3	$\cos\delta_4=\sin x_4$	0,99500	0,99578	0,99576
4	$0,21212/(3)$	0,21319	0,21302	0,21302
5	$\mathrm{ch}\,y_4=(4)\cdot(2)$	5,69079	5,66880	5,66876
6	$y_4=\mathrm{arch}\,(5)$	2,42419	2,42025	2,42024
7	$\mathrm{sh}\,y_4$	5,60229	5,57990	5,57984
8	$(5)^2-(7)^2$	0,99944	1,00001	1,00022
9	$0,21212\cdot(6)$	0,51422	0,51338	0,51338
10	$\sin\delta_4=\cos x_4=\frac{(9)}{(7)}$	0,09179	0,09201	0,09201
11	$(3)^2+(10)^2$	0,99845	1,00004	1,00000
12	$\delta_4=\arcsin\,(10)$	0,09192	0,09214	0,09214

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Everywhere in the equations, where enters Poisson ratio, we set/assume him equal to 0.3:

1) the equation

$$\sin z = \frac{1-\mu}{3+\mu} z, \quad \text{i. e.} \quad \sin z = 0{,}21212z,$$

$$\left.\begin{aligned}
z_0 &= 2.56603;\\
z_1 &= 7.67774 + 1.08927i,\\
z_2 &= 14.00454 + 1{,}76167i,\\
z_3 &= 20.31217 + 2{,}14605i,\\
z_4 &= 26.61139 + 2.42024i,\\
z_5 &= 32{,}90601 + 2{,}63431i;
\end{aligned}\right\} \qquad (5.49)$$

2) the equation

$$\sin z = \frac{1+\mu}{3-\mu} z, \quad \text{i.e.} \quad \sin z = 0{,}48148z,$$

$$\left.\begin{aligned}
z_0 &= 1{,}93827;\\
z_1 &= 7{,}58626 + 2{,}00697i,\\
z_2 &= 13{,}95025 + 2{,}60992i,\\
z_3 &= 20{,}27368 + 2{,}97980i,\\
z_4 &= 26{,}58162 + 3{,}24842i,\\
z_5 &= 32{,}88169 + 3{,}45973i;
\end{aligned}\right\} \qquad (5.50)$$

3)

$$\sin z = -\frac{1+\mu}{3-\mu} z, \text{ i. e. } \sin z = +0{,}48148z;$$

$$\left.\begin{aligned} z_1 &= 4{,}35754 + 1{,}44443i, \\ z_2 &= 10{,}77666 + 2{,}35485i, \\ z_3 &= 17{,}11475 + 2{,}81210i, \\ z_4 &= 23{,}42889 + 3{,}12319i, \\ z_5 &= 29{,}73234 + 3{,}35967i; \end{aligned}\right\} \quad (5.51)$$

TABLE 5.

$s=5;\ \frac{\pi}{2}(4s+1)=\frac{21\pi}{2}=32{,}98672$				
1	δ_s	0,07	0,08074	0,08071
2	$x_s = 32{,}98672 - \delta_s$	32,91672	32,90598	32,90601
3	$\cos\delta_s = \sin x_s$	0,99755	0,99674	0,99674
4	0,21212/(3)	0,21264	0,21281	0,21281
5	$\operatorname{ch} y_s = (4)\cdot(2)$	6,99941	7,00272	7,00273
6	$y_s = \operatorname{arch}(5)$	2,63383	2,63431	2,63431
7	$\operatorname{sh} y_s$	6,92761	6,93097	6,93097
8	$(5)^2 - (7)^2$	0,99996	0,99974	0,99988
9	0,21212·(6)	0,55869	0,55879	0,55879
10	$\sin\delta_s = \cos x_s = \frac{(9)}{(7)}$	0,08065	0,08062	0,08062
11	$(3)^2 + (10)^2$	1,00161	0,99999	0,99999
12	$\delta_s = \arcsin(10)$	0,08074	0,08071	0,08071

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4) the equation

$$\sin z = z,$$

$$\left.\begin{aligned} z_1 &= 7.49767 + 2.76867i, \\ z_2 &= 13.89995 + 3.35220i, \\ z_3 &= 20.23871 + 3.71676i, \\ z_4 &= 26.55454 + 3.98314i, \\ z_5 &= 32.85974 + 4.19325i; \end{aligned}\right\} \quad (5.52)^*$$

5) the equation

$$\sin z = -z,$$

$$\left.\begin{aligned} z_1 &= 4.21239 + 2.25072i, \\ z_2 &= 10.71253 + 3.10314i, \\ z_3 &= 17.07336 + 3.55108i, \\ z_4 &= 23.39835 + 3.85880i, \\ z_5 &= 29.70811 + 4.09370i; \end{aligned}\right\} \quad (5.53)^*$$

6) the equation

$$(3-\mu)(1+\mu)\cos z - \frac{1}{2}(1+\mu)^2 z^2 + 4 + (1-\mu)^2 = 0.$$

*These roots are borrowed from work to Fadle[47].

i.e.

$$\left.\begin{aligned}
&5.31261 + 4.15385\cos z - z^2 = 0,\\
&z_0 = 1{,}94653\\
&z_1 = 5{,}17172 + 2{,}71980i\\
&z_2 = 11{,}84600 + 4.31180i\\
&z_3 = 18{,}29300 + 5{,}14570i\\
&z_4 = 24.67304 + 5.72550i\\
&z_5 = 31{,}02110 + 6.17256i;
\end{aligned}\right\} \quad (5.54)$$

7) the equation

$$(3+\mu)(1-\mu)\cos z - \frac{1}{2}(1-\mu)^2 z^2 + 4 + (1+\mu)^2 = 0,$$

i.e.

$$\left.\begin{aligned}
&23.2245 + 9.42856\cos z - z^2 = 0;\\
&z_0 = 5{,}41366,\\
&z_1 = 4.0545 + 0.71300i\\
&z_2 = 11{,}92765 + 3.34835i.
\end{aligned}\right\} \quad (5.55)$$

* These roots are borrowed from work to Padle [47].

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§25. Rigidly embedded rectangular plate.

In this and the following paragraphs are examined the specific problems of the applied theory of elasticity. In this case the expressed goal to so much not obtain the complete (in engineering sense) solution of precisely these problems, as to illustrate common/general/total course of solution by the method of the initial functions. Therefore are given only some numerical data, necessary for the engineering. In particular this is related to §26 and 27. As concerns the selection of problems themselves, it is accidental.

Is examined the rigidly embedded on outline/contour rectangular plate (Fig. 23), on which acts evenly distributed load p.

On the strength of the symmetry of strain is selected axle/axis, as shown in figure. Then in general solution (4.93) one should assume

$$\theta_0 = V_0 = 0. \qquad (5.56)$$

a instead of the particular solution w_p, θ_{xp} and so forth to take expressions (4.96). Then

$$\left.\begin{aligned}
w(\xi, \eta) &= a\left\{L_{ww}W_0 + L_{wM}M_0 + \frac{pa^3}{24D}\xi^4\right\},\\
\theta_x(\xi, \eta) &= L_{\theta_x w}W_0 + L_{\theta_x M}M_0,\\
\theta_y(\xi, \eta) &= L_{\theta w}W_0 + L_{\theta M}M_0 + \frac{pa^3}{6D}\xi^3,\\
M_x(\xi, \eta) &= \frac{D}{a}\left\{L_{M_x w}W_0 + L_{M_x M}M_0 - \frac{\mu pa^3}{2D}\xi^2\right\},\\
M_y(\xi, \eta) &= \frac{D}{a}\left\{L_{Mw}W_0 + L_{MM}M_0 - \frac{pa^3}{2D}\xi^2\right\},\\
V_x(\xi, \eta) &= \frac{D}{a^3}\left\{L_{Vw}W_0 + L_{VM}M_0 - \frac{pa^2}{D}\xi\right\},
\end{aligned}\right\}$$

$$\left.\begin{aligned}
V_y(\xi, \eta) &= \frac{D}{a^3}\{L_{V_y w}W_0 + L_{V_y M}M_0\},\\
R(\xi, \eta) &= \frac{D}{a}(1-\mu)\{L_{Rw}W_0 + L_{RM}M_0\}.
\end{aligned}\right\} \quad (5.57)$$

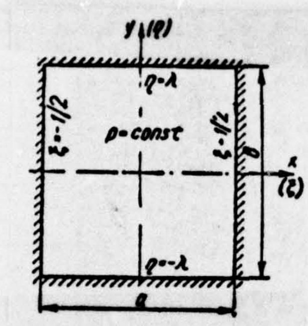

Fig. 23.

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Recall that

$$\xi = \frac{x}{a}, \quad \eta = \frac{y}{a}, \quad \lambda = \frac{b}{a}. \tag{5.58}$$

We satisfy the conditions of the attachment

$$w\left(\frac{1}{2}, \eta\right) = \theta_y\left(\frac{1}{2}, \eta\right) = 0 \tag{5.59}$$

at the right edge of the plate

$$\left.\begin{aligned} L_{ww}\left(\frac{1}{2}, \beta\right) W_0(\eta) + L_{wM}\left(\frac{1}{2}, \beta\right) M_0(\eta) &= -\frac{pa^3}{384D}, \\ L_{\theta w}\left(\frac{1}{2}, \beta\right) W_0(\eta) + L_{\theta M}\left(\frac{1}{2}, \beta\right) M_0(\eta) &= -\frac{pa^3}{48D}. \end{aligned}\right\} \quad (5.60)$$

For the determination of the particular solution of this system is decomposed the entering here operators series (they easily are obtained from (4.100) and (4.101)

$$\left.\begin{aligned} &\left(1 - \frac{\mu}{8}\frac{d^2}{d\eta^2} + \ldots\right) W_0(\eta) + \\ &+\left(-\frac{1}{8} + \frac{1}{192}\frac{d^2}{d\eta^2} - \ldots\right) M_0(\eta) = -\frac{pa^3}{384D}, \\ &\left(-\frac{\mu}{2}\frac{d^2}{d\eta^2} - \ldots\right) W_0(\eta) + \\ &+\left(-\frac{1}{2} + \frac{1}{48}\frac{d^2}{d\eta^2} - \ldots\right) M_0(\eta) = -\frac{pa^3}{48D}. \end{aligned}\right\} \quad (5.61)$$

Hence

$$w_{0\,\text{част}} = \frac{pa^3}{384D}, \quad M_{0\,\text{част}} = \frac{pa^3}{24D}. \qquad (5.62)$$

The general solution of uniform system (5.60) and the resolving equation for the present instance can be undertaked from (5.29). After taking, expressions for the necessary operators from (4.100) and (4.101), we will obtain thus

$$\left.\begin{aligned} w_0(\eta) &= \frac{pa^4}{D}\left[\frac{1}{384}+\frac{\sin\beta/2}{\beta}\varphi(\eta)\right], \\ M_0(\eta) &= \frac{pa^2}{D}\left\{\frac{1}{24}+\left[4\cos\frac{\beta}{2}+(1-\mu)\beta\sin\frac{\beta}{2}\right]\varphi(\eta)\right\}, \end{aligned}\right\} \quad (5.63).$$

where $\varphi(\eta)$ it satisfies the equation

$$\left(1+\frac{\sin\beta}{\beta}\right)\varphi(\eta)=0. \quad (5.64)$$

Pages 176-177. Pages missing.

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$$= -\frac{(1-4\xi^2)^2}{384}, \quad (5.77)$$

$$\operatorname{Re}\sum_{n=1}^{\infty}\left\{(a_n+ib_n)\sin\frac{k_n}{2}\cos k_n\xi-2\cos\frac{k_n}{2}\cdot\xi\sin k_n\xi)\operatorname{sh}\frac{k_n\lambda}{2}\right\}=0. \quad (5.78)$$

To find the hence precise values of coefficients a_n and b_n scarcely is possible. In order to obtain approximate values for them, is decomposed the functions, entering equations (5.77) and (5.78), in series according to degrees ξ, we will be bounded in expansion only by first member let us equate zero coefficients with this member. For calculations let us assume λ = 1.2, μ = 0.3 and preliminarily let us find

$$\left.\begin{aligned} \sin\frac{k_1}{2} &= 1,46460-0,70319i, \\ \cos\frac{k_1}{2} &= -0,86880-1,18542i, \\ \operatorname{ch}\frac{k_1\lambda}{2} &= 1,37725+6,07029i. \end{aligned}\right\} \qquad (5.79)$$

Then from (5.75) we obtain for the characteristic points of the plate: in center $(\xi=0,\ \eta=0)$

$$\begin{aligned} w_0 &= \frac{pa^4}{384D}(1+77,219a_1+105,36b_1), \\ M_{x_0} &= \frac{pa^2}{24}(0,3-155,26a_1+39,756b_1), \\ M_{y_0} &= \frac{pa^2}{24}(1+46,831a_1+108,18b_1); \end{aligned} \qquad (5.80)$$

at point $\left(\xi = \frac{1}{2},\ \eta = 0\right)$

$$M_y = \frac{pa^2}{24}(-2 - 62{,}438a_1 - 197{,}74b_1); \qquad (5.81)$$

at point $\left(\xi = 0,\ \eta = \frac{\lambda}{2}\right)$

$$M_x = \frac{pa^2}{24}(0{,}3 + 27{,}502a_1 + 997{,}21b_1). \qquad (5.82)$$

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For determining a_1 and b_1, we record/write system of equations, which are obtained from the condition of reduction to zero only absolute terms in expansions w and θ_y according to degrees ξ:

$$\left.\begin{aligned} 1{,}94252a_1 - 0{,}842786_1 &= -\frac{1}{384}, \\ 6{,}28569a_1 - 7{,}922086_1 &= 0. \end{aligned}\right\} \qquad (5.83)$$

Hence

$$a_1 = -0.0020444, \quad b_1 = -0.0016221. \qquad (5.84)$$

By substituting these values in (5.80) - (5.82), we will obtain in the center of the plate

$$\left.\begin{aligned} w &= 0{,}0191\frac{pa^4}{E\delta^3}\,(0{,}0188;\ 0{,}0191), \\ M_{x_0} &= 0{,}0230pa^2\,(0{,}0228;\ 0{,}0231), \\ M_{y_0} &= 0{,}0304pa^2\,(0{,}0299;\ 0{,}0302); \end{aligned}\right\} (5.85)$$

the supporting moment of the halfway long side

$$M_y = -0{,}0646pa^2\,(-0{,}0639;\ -0{,}0612); \quad (5.86)$$

the supporting moment of the halfway short side

$$M_y = -0{,}0572pa^2\,(-0{,}0554;\ -0{,}0504). \quad (5.87)$$

Here in brackets for a comparison are given values, given by S.

P. Timoshenko [40, page 222] and by [redacted]. V. and by Ye. D. Vaynbergs [6, pages 452].

§26. Rectangular plate-arm.

Retaining in force introduction to §25, let us note, that since the solution here will be carried out according to the same plan/layout, as in §25, the intermediate lining/calculations and explanations will be considerably abbreviated/reduced.

Is examined the evenly loaded cantilever plate (Fig. 24).

Let us select the initial line in stopping up, and particular solution again let us take in the form (4.96).

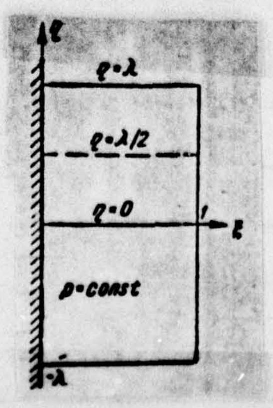

Fig. 24.

Fig. 24.

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Then it will be:

$$
\left.\begin{aligned}
w(\xi,\eta) &= a\left\{L_{wM}M_0 + L_{wV}V_0 + \frac{pa^4}{24D}\xi^4\right\},\\
\theta_x(\xi,\eta) &= L_{\theta_x M}M_0 + L_{\theta_x V}V_0,\\
\theta_y(\xi,\eta) &= L_{\theta M}M_0 + L_{\theta V}V_0 + \frac{pa^3}{6D}\xi^3,\\
M_x(\xi,\eta) &= \frac{D}{a}\left\{L_{M_x M}M_0 + L_{M_x V}V_0 - \mu\frac{pa^3}{2D}\xi^2\right\}.
\end{aligned}\right| \quad (5.88)
$$

$$
\left.\begin{aligned}
M_y(\xi,\eta) &= \frac{D}{a}\left\{L_{MM}M_0 + L_{MV}V_0 - \frac{pa^3}{2D}\xi^2\right\},\\
V_x(\xi,\eta) &= \frac{D}{a^2}\left\{L_{VM}M_0 + L_{VV}V_0 - \frac{pa^3}{D}\xi\right\},\\
V_y(\xi,\eta) &= \frac{D}{a^2}\left\{L_{V_y M}M_0 + L_{V_y V}V_0\right\},\\
R(\xi,\eta) &= \frac{D}{a}(1-\mu)\{L_{RM}M_0 + L_{RV}V_0\}.
\end{aligned}\right| \quad (5.88)
$$

At edge $\xi = 1$

$$
\left.\begin{aligned}
&L_{MM}(1, \beta)\, M_0(\eta) + L_{MV}(1, \beta)\, V_0(\eta) = \frac{pa^2}{2D}, \\
&L_{VM}(1, \beta)\, M_0(\eta) + L_{VV}(1, \beta)\, V_0(\eta) = \frac{pa^2}{D}.
\end{aligned}\right\} \qquad (5.89)
$$

For the determination of particular solution, taking into account (4.100) and (4.101), we have

$$
\left.\begin{aligned}
&\left(1 - \frac{2-\mu}{2}\beta^2 - \frac{3-2\mu}{24}\beta^4 - \ldots\right)\widetilde{M}_0 + \left(1 - \frac{2-\mu}{6}\beta^2 + \right. \\
&\qquad \left. + \frac{3-2\mu}{120}\beta^4 - \ldots\right)\widetilde{V}_0 = \frac{pa^2}{2D}, \\
&\left(-\mu\beta^2 - \frac{1-2\mu}{6}\beta^4 + \ldots\right)\widetilde{M}_0 + \left(1 - \frac{\mu}{2}\beta^2 - \right. \\
&\qquad \left. - \frac{1-2\mu}{24}\beta^4 + \ldots\right)\widetilde{V}_0 = \frac{pa^2}{D}.
\end{aligned}\right\} \qquad (5.90)
$$

Hence

$$
\widetilde{M}_0 = -\frac{pa^2}{2D}, \quad \widetilde{V}_0 = \frac{pa^2}{D}. \qquad (5.91)
$$

a keeping in mind (5.36).

$$\left.\begin{aligned} M_0 &= -\frac{pa^2}{2D} - L_{MV}(1, \beta)\,\varphi(\eta), \\ V_0 &= \frac{pa^2}{D} + L_{MM}(1, \beta)\,\varphi(\eta), \end{aligned}\right\} \tag{5.92}$$

cr

$$\left.\begin{aligned} M_0 &= -\frac{pa^2}{2D} - \left(\frac{1+\mu}{2}\frac{\sin\beta}{\beta} + \frac{1-\mu}{2}\cos\beta\right)\varphi(\eta), \\ V_0 &= \frac{pa^2}{D} + \left(\cos\beta - \frac{1+\mu}{2}\beta\sin\beta\right)\varphi(\eta). \end{aligned}\right\} \tag{5.93}$$

where $\varphi(\eta)$ satisfies the equation

$$\left[1 - \frac{5+2\mu+\mu^2}{2} - \frac{3-2\mu-\mu^2}{2}\cos 2\beta + \frac{(1-\mu)^2}{4}(2\beta)^2\right]\varphi(\eta) = 0. \tag{5.94}$$

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By characteristic equation with $\mu = 0.3$ and $z = 2k$ will be

$23.2245 + 9.42856 \cos z - z^2 = 0$. According to (5.55) its roots

$$\left.\begin{aligned} k_0 &= 2{,}70683, \\ k_1 &= 2{,}02727 + 0{,}35650i, \\ k_2 &= 5{,}96382 + 1{,}67418i. \end{aligned}\right\} \qquad (5.96)$$

In accordance with (5.94) and (5.96), and also with symmetry relative tc axle/axis ξ

$$\varphi(\eta) = \frac{pa^4}{D}\left\{A_0 \operatorname{ch} k_0\eta + 2\operatorname{Re}\sum_{n=1}^{\infty}(a_n + ib_n)\operatorname{ch} k_n\eta\right\}. \qquad (5.97)$$

Now from (5.88), (5.93) and (5.97), accepting $\mu = 0.3$ and substituting concrete/specific/actual expressions for the operators of (4.100) and (4.101), we will obtain

$$\frac{D}{pa^4}w(\xi,\eta)=\frac{\xi^2(6-4\xi+\xi^2)}{24}+A_0\psi_w(\xi,k_0)\,\mathrm{ch}\,k_0\eta+$$
$$+2\mathrm{Re}\sum_{n=1}^{\infty}(a_n+ib_n)\,\psi_w(\xi,k_n)\,\mathrm{ch}\,k_n\eta,$$
$$\frac{1}{pa^2}M_x(\xi,\eta)=-0{,}15(1-\xi)^2+A_0\psi_{M_x}(\xi,k_0)\,\mathrm{ch}\,k_0\eta+$$
$$+2\mathrm{Re}\sum_{n=1}^{\infty}(a_n+ib_n)\,\psi_{M_x}(\xi,k_n)\,\mathrm{ch}\,k_n\eta,$$
$$\frac{1}{pa^2}M_y(\xi,\eta)=-\frac{(1-\xi)^2}{2}+A_0\psi_{M_y}(\xi,k_0)\,\mathrm{ch}\,k_0\eta+ \tag{5.98}$$
$$+2\mathrm{Re}\sum_{n=1}^{\infty}(a_n+ib_n)\,\psi_{M_y}(\xi,k_n)\,\mathrm{ch}\,k_n\eta,$$
$$\frac{1}{pa}V_x(\xi,\eta)=1-\xi+A_0\psi_{V_x}(\xi,k_0)\,\mathrm{ch}\,k_0\eta+$$
$$+2\mathrm{Re}\sum_{n=1}^{\infty}(a_n+ib_n)\,\psi_{V_x}(\xi,k_n)\,\mathrm{ch}\,k_n\eta,$$

$$\frac{1}{pa}V_y(\xi,\eta)=A_0\psi_{V_y}(\xi,k_0)\,\mathrm{sh}\,k_0\eta+$$
$$+2\mathrm{Re}\sum_{n=1}^{\infty}(a_n+ib_n)\,\psi_{V_y}(\xi,k_n)\,\mathrm{sh}\,k_n\eta,$$
$$\frac{1}{pa^2}\frac{R(1,\eta)}{1-\mu}=A_0\psi_R(1,k_0)\,\mathrm{sh}\,k_0\eta+ \tag{5.98}$$
$$+2\mathrm{Re}\sum_{n=1}^{\infty}(a_n+ib_n)\,\psi_R(1,k_n)\,\mathrm{sh}\,k_n\eta.$$

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Here

$$\left.\begin{aligned}
&\psi_w(\xi, k) = -\frac{0,5}{k^3}\varkappa(k)\sin k\xi + \frac{1,42857}{k^3}\varrho(\xi, k),\\
&\psi_{M_x}(\xi, k) = -0,15\gamma(k)\cos k\xi + \frac{0,65}{k}\varkappa(k)\sin k\xi -\\
&\quad -0,35\varkappa(k)\,\xi\cos k\xi - 0,175k\gamma(k)\,\xi\sin k\xi,\\
&\psi_{M_y}(\xi, k) = -0,5\gamma(k)\cos k\xi + \frac{0,65}{k}\varkappa(k)\sin k\xi + \varrho(\xi, k),\\
&\psi_{V_x}(\xi, k) = \varkappa(k)\cos k\xi + 0,325k\gamma(k)\sin k\xi -\\
&\quad -0,175\xi k\,[k\gamma(k)\cos k\xi - 2\varkappa(k)\sin k\xi],\\
&\psi_{V_y}(\xi, k) = -0,85k\gamma(k)\cos k\xi + 1,35\varkappa(k)\sin k\xi +\\
&\quad +0,35k\varkappa(k)\,\xi\cos k\xi + 0,175k^2\gamma(k)\,\xi\sin k\xi,\\
&\psi_R(1, k) = -0,325\frac{\sin^2 k}{k} - 0,175k.
\end{aligned}\right\}\quad(5.99)$$

a

$$\left.\begin{aligned}
&\gamma(k) = (1+\mu)\frac{\sin k}{k} + (1-\mu)\cos k = 1,3\frac{\sin k}{k} + 0,7\cos k,\\
&\varkappa(k) = \cos k - \frac{1-\mu}{2}k\sin k = \cos k - 0,35\,k\sin k,\\
&\varrho(\xi, k) = 0,175\,[2\varkappa(k)\cos k\xi + k\gamma(k)\sin k\xi].
\end{aligned}\right\}\quad(5.100)$$

By satisfying conditions at edges $\eta=\pm\lambda$ and in free angles, let us arrive at the following:

$$-0.15(1-\xi)^3+A_0\psi_{M_\xi}(\xi,k_0)\,\mathrm{ch}\,k_0\lambda+$$

$$+2\mathrm{Re}\sum_{n=1}^{\infty}(a_n+ib_n)\psi_{M_\xi}(\xi,k_n)\,\mathrm{ch}\,k_n\lambda=0. \qquad (5.101)$$

$$A_0\psi_{V_\xi}(\xi,k_0)\,\mathrm{sh}\,k_0\lambda+2\mathrm{Re}\sum_{n=1}^{\infty}(a_n+ib_n)\psi_{V_\xi}(\xi,k_n)\,\mathrm{sh}\,k_n\lambda=0. \qquad (5.102)$$

$$A_0\psi_R(1,k_0)\,\mathrm{sh}\,k_0\lambda+2\mathrm{Re}\sum_{n=1}^{\infty}(a_n+ib_n)\psi_R(1,k_n)\,\mathrm{sh}\,k_n\lambda=0. \qquad (5.103)$$

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Let us assume for a concreteness $\lambda = 1$ (plate with the relation of sides 1:2). Equation (5.103) can been satisfied accurately, in the relation to equations (5.101) and (5.102) this is extremely difficult or generally impossibly. Therefore let us act as follows: we will be bounded in series $n = 1$, is decomposed function $\psi_{M_x}(\xi, k)$ and $\psi_{V_y}(\xi, k)$ in series according to degrees ξ

$$\left.\begin{aligned}\psi_{M_x}(\xi, k) &= -0{,}15\gamma(k) + 0.3\varkappa(k)\xi - 0{,}1k^2\gamma(k)\xi^2 + \ldots\\ \psi_{V_y}(\xi, k) &= -0{,}85k\gamma(k) + 1{,}7k\varkappa(k) - 0{,}60k^2\gamma(k)\xi^2 - \ldots\end{aligned}\right\} \quad (5.104)$$

and let us equate in equations (5.101) and (5.102) absolute terms zero, a (5.103) let us write accurately. This will lead to the system

$$\left.\begin{aligned}-3{,}25491A_0 + 2{,}92518a_1 + 2{,}48870b_1 &= -0.5\\ -8{,}73235A_0 + 6{,}72762a_1 + 3{,}74716b_1 &= 0,\\ -3{,}69146A_0 - 3{,}41204a_1 + 1.23890b_1 &= 0.\end{aligned}\right\} \quad (5.105)$$

Whence

$$A_0 = -0{,}18084, \quad a_1 = +0{,}02581, \quad b_1 = -0{,}46776. \quad (5.106)$$

Let us make table of auxiliary values (Table 6-8).

TABLE 6.

ξ	$\varkappa_0\xi$	$\varkappa_1\xi$	$\varkappa_2\xi$
0	0	0	0
0,2	+0,54137	+0,40545 + 0,07130 i	+1,19276 + 0,33484 i
0,4	+0,108273	+0,81091 + 0,14260 i	+2,38553 + 0,66967 i
0,6	+1,62410	+1,21636 + 0,21390 i	+3,57829 + 1,00451 i
0,8	+2,16546	+1,62182 + 0,28520 i	+4,77106 + 1,33934 i
1	+2,70683	+2,02727 + 0,35650 i	+5,96382 + 1,67418 i

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$\mathrm{ch}\, k_0 \frac{\lambda}{2} = +2,06445$	$\mathrm{sh}\, k_0 \frac{\lambda}{2} = +1,80612$
$\mathrm{ch}\, k_1 \frac{\lambda}{2} = +1,53453 + 0,21212i$	$\mathrm{sh}\, k_1 \frac{\lambda}{2} = +1,17737 + 0,27647i$
$\mathrm{ch}\, k_2 \frac{\lambda}{2} = +6,62160 + 7,30551i$	$\mathrm{sh}\, k_2 \frac{\lambda}{2} = +6,58666 + 7,34316i$
$\mathrm{ch}\, k_0\lambda = 7,52424$	$\mathrm{sh}\, k_0\lambda = 7,45749$
$\mathrm{ch}\, k_1\lambda = 3,61963 + 1,30206i$	$\mathrm{sh}\, k_1\lambda = 3,49624 + 1,34801i$
$\mathrm{ch}\, k_2\lambda = 20,0774 + 193,507i$	$\mathrm{sh}\, k_2\lambda = 20,0771 + 193,509i$

Using these data, we compute first of all values M_x and V_y at the free edges, where they must be equal to zero (Table 9).

For M_{max} and $V_{y\,max}$ we take greatest the bending moment and transverse force in the beam-strip, which corresponds to the plate, i.e., M_{max} in question $=-0.5pa^2$, a $V_{y\,max} = pa$.

Table 7.

ξ	$\sin \kappa_{0}\xi$	$\sin \kappa_{1}\xi$	$\sin \kappa_{2}\xi$
0	0	0	0
0,2	+0,51531	+0,39543 + 0,065574 i	+0,98198 + 0,12591 i
0,4	+0,88324	+0,73231 + 0,098566 i	+0,84573 — 0,52445 i
0,6	+0,99858	+0,95938 + 0,074805 i	—0,65490 — 1,07122 i
0,8	+0,82833	+1,03960 — 0,014714 i	—2,03576 + 0,10422 i
1	+0,42121	+0,95525 — 0,16049 i	—0,86682 + 2,44336 i

Table 8.

ξ	$\cos \kappa_{0}\xi$	$\cos \kappa_{1}\xi$	$\cos \kappa_{2}\xi$
0	1	1	1
0,2	+0,85700	+0,92125 — 0,028146 i	+0,38999 — 0,31704 i
0,4	+0,46891	+0,69586 — 0,10373 i	—0,89686 — 0,49455 i
0,6	—0,05328	+0,35503 — 0,20214 i	—1,40308 + 0,50000 i
0,8	—0,56023	—0,052984 — 0,28870 i	+0,11958 + 1,77419 i
1	—0,90697	—0,46909 — 0,32682 i	+2,62135 + 0,80797 i

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Table 9 shows that at values of coefficients (5.106) the solution is obtained satisfactorily. Introducing (5.106) and (5.99) in (5.98), we compute amounts of deflection and bending moments in a series of the points of the plates, necessary for construction diagram/curve. Table 10 and 11 for a comparison give values $\frac{D}{pa^4}w(\xi,\lambda)$ $\frac{1}{pa^2}M_y(\xi,\lambda)$ respectively, given P. M. Varvakam [8] for ξ = 0; 0.25; 0.5; 0.75; 1.

Table 9.

ξ	$\frac{1}{pa^3}M_x(\xi,\lambda)$	M_{max}, %	$\frac{1}{pa}V_y(\xi,\lambda)$	$V_{y\,max}$, %
0	−0,00001	0	−0,00003	0
0,2	−0,018	3,6	−0,062	6,2
0,4	−0,023	4,6	−0,031	3,1
0,6	−0,015	3,0	+0,023	2,3
0,8	−0,001	0,2	+0,018	1,8
1	+0,00002	0	+0,00002	0

Table 10.

ξ	$\frac{D}{pa^4}w(\xi,0)$	$\frac{D}{pa^4}w\left(\xi,\frac{\lambda}{2}\right)$	$\frac{D}{pa^4}w(\xi,\lambda)$	
			(1) по нашим данным	(2) по данным П. М. Варвака
0	0,0000	0,0000	0,0000	0
0,2	0,0084	0,0081	0,0064	
0,4	0,0330	0,0356	0,0399	0,0127
0,6	0,0627	0,0660	0,0842	0,0433
0,8	0,0927	0,0937	0,0992	0,0823
1	0,1215	0,1180	0,0987	0,1235

Key: (1). according to our data. (2). according to data of P. M. Varvak.

Table 11.

ξ	$\frac{1}{pa^3}M_y(\xi,0)$	$\frac{1}{pa^3}M_y\left(\xi,\frac{\lambda}{2}\right)$	$\frac{1}{pa^3}M_y(\xi,\lambda)$	
			(1) по нашим данным	(2) по данным П. М. Варвака
0	−0,5035	−0,5070	−0,5263	−0,4640
0,2	−0,3218	−0,3236	−0,3335	
0,4	−0,1794	−0,1788	−0,1755	−0,2812
0,6	−0,0780	−0,0760	−0,0650	−0,1286
0,8	−0,0198	−0,0196	−0,0185	−0,0317
1	0,0000	0,0000	0,0000	0

Key: (1). according to our data. (2). according to data of P. M. Varvak.

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According to these data on Fig. 25 are constructed diagram/curves w and M_ξ in to section $\lambda = 0;\ \lambda/2;\ \lambda$.

Recall that boundary conditions at edges $\xi = 0$ and $\xi = 1$, and also in angles $(1, \pm\lambda)$ are carried out accurately. If we in series (5.98) select $n > 1$, then in principle it is possible more accurately to satisfy conditions (5.101) and (5.102). However, in practice, by increasing n, it is possible to arrive also at the opposite result: accuracy/precision is reduced. This is connected with the fact that, as can be seen from (5.96), the module/moduli of roots grow/rise with an increase in the index. In this case especially intensely grow/rise and converge between themselves $ch\kappa\lambda$ and $sh\kappa\lambda$ (see page 185). Therefore, by increasing n, it is necessary to simultaneously increase a quantity of signs in numbers. For an example we can indicate that was carried out the arithmetic count with an accuracy to 5-6 signs. For five unknowns was obtained the system

$$\left.\begin{aligned}
&-3{,}25491A_0+2{,}92518a_1+2{,}48870b_1-\\
&\qquad-501{,}200a_2-651{,}488b_2=-0{,}5,\\
&-3{,}90666A_0-4{,}06398a_1+1{,}72688b_1-\\
&\qquad-162{,}973a_2+268{,}070b_2=0,\\
&-8{,}73235A_0+6{,}72762a_1+3{,}74716b_1-\\
&\qquad-4079{,}80a_2-3046{,}30b_2=0.\\
&-5{,}64360A_0-3{,}93864a_1+2{,}68626b_1-\\
&\qquad-281{,}712a_2+1007{,}78b_2=0,\\
&-3{,}69146A_0-3{,}41204a_1+1{,}23890b_1+\\
&\qquad+88{,}1248a_2+272{,}510b_2=0.
\end{aligned}\right\} \tag{5.107}$$

Its roots

$$\left.\begin{aligned}
&A_0=-0{,}036341,\\
&a_1=-0{,}037356\quad a_2=-0{,}00031154,\\
&b_1=-0{,}22474\quad b_2=+0{,}00016245,
\end{aligned}\right\} \tag{5.108}$$

they differ significantly from the same of system (5.106). Constructed according to these data curve/graph $D/pa^4\ w\ (\xi,\lambda)$ (dotted line on Fig. 25) shows that the picture of sagging/deflections, and means and effort/forces and torque/moments were not improved, but it

deteriorated.

§27. Cantilever beam.

As one additional illustration let us consider curvature by force P of the cantilever beam of the rectangular cross section h x t (Fig. 26) and explain the distribution of normal stresses over the pinched section.

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Let us for simplicity consider that as usually in strength of materials, the load is distributed on end/face according to parabolic law. For distribution σ over the embedded section, this fact does not affect, since the beam is chosen by sufficiently long.

Let us assume

$$\xi=\frac{x}{l},\quad \eta=\frac{y}{l},\quad \lambda=\frac{h}{l} \qquad (5.109)$$

and write the boundary conditions:

$$u|_{\xi=0} = v|_{\xi=0} = 0. \quad (5.110)$$

$$\sigma|_{\xi=1} = 0, \quad \tau_{xy}'|_{\xi=1} = -\frac{3}{2}\frac{P}{ht}\left(1 - \frac{4\eta^2}{\lambda^2}\right). \quad (5.111)$$

$$\sigma_y|_{\eta=\pm\frac{\lambda}{2}} = \tau_{xy}|_{\eta=\pm\frac{\lambda}{2}} = 0. \quad (5.112)$$

The initial line is consistent with the pinched section. Then, according to (5.110),

$$U = V_0 = 0 \quad (5.113)$$

and, on (4.62),

$$\left.\begin{aligned} u(\xi, \eta) &= l(L_{u\sigma}\sigma_0 + L_{u\tau}\tau_0), \\ v(\xi, \eta) &= l(L_{v\sigma}\sigma_0 + L_{v\tau}\tau_0), \\ \sigma_x(\xi, \eta) &= G(L_{\sigma\sigma}\sigma_0 + L_{\sigma\tau}\tau_0), \\ \sigma_y(\xi, \eta) &= G(L_{y\sigma}\sigma_0 + L_{y\tau}\tau_0), \\ \tau_{xy}(\xi, \eta) &= G(L_{\tau\sigma}\sigma_0 + L_{\tau\tau}\tau_0). \end{aligned}\right\} \quad (5.114)$$

Factors 1 and G are introduced in order that the brackets would be dimensionless. We satisfy conditions (5.111)

$$\left.\begin{aligned} &L_{\sigma\sigma}(1,\beta)\sigma_0(\eta)+L_{\sigma\tau}(1,\beta)\tau_0(\eta)=0,\\ &L_{\tau\sigma}(1,\beta)\sigma_0(\eta)+L_{\tau\tau}(1,\beta)\tau_0(\eta)=-\frac{3}{2}\frac{P}{ht}\left(1-\frac{4\eta^2}{\lambda^2}\right).\end{aligned}\right\} \quad (5.115)$$

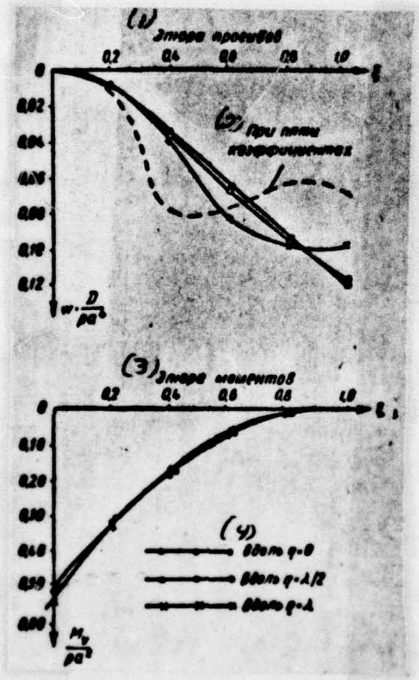

Fig. 25.

Key: (1). Diagram of deflections. (2). With five coefficients. (3). Diagram of torque/moments. (4). lengthwise ....

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Hence

$$\left.\begin{aligned}&\sigma_0(\eta) = -L_{\sigma\tau}(1,\beta)\,\varphi(\eta),\\&\tau_0(\eta) = L_{\sigma\sigma}(1,\beta)\,\varphi(\eta),\\&[L_{\sigma\sigma}(1,\beta)\,L_{\tau\tau}(1,\beta) - L_{\sigma\tau}(1,\beta)\,L_{\tau\sigma}(1,\beta)]\,\varphi(\eta) = -\frac{3}{2}\frac{P}{ht}\left(1-\frac{4\eta^2}{\lambda^2}\right);\end{aligned}\right\} \tag{5.116}$$

or, if we substitute expressions for the operators from (4.83),

$$\left.\begin{aligned}&\sigma_0(\eta) = \left(+\frac{1-\mu}{2}\sin\beta + \frac{1+\mu}{2}\beta\cos\beta\right)\varphi(\eta),\\&\tau_0(\eta) = \left(+\cos\beta + \frac{1+\mu}{2}\beta\sin\beta\right)\varphi(\eta)\end{aligned}\right\}, \tag{5.117}$$

and

$$\left[5-2\mu+\mu^2+(3+2\mu-\mu^2)\cos 2\beta - \frac{(1+\mu)^2}{2}(2\beta)^2\right]\varphi(\eta) = -\frac{12P}{ht}\left(1-\frac{4\eta^2}{\lambda^2}\right). \tag{5.118}$$

Fig. 25.

$$\varphi_{\text{част}} = \frac{6P}{\lambda^2 htG}\left[2(1+\mu) - \frac{\lambda^2}{4} + \eta^2\right]. \qquad (5.119)$$

The particular solution to the last/latter equation

a homogeneous equation (5.118) reduces to characteristic equation (5.54); therefore $\varphi(\eta)$ equally

$$\varphi(\eta) = \frac{6P}{\lambda^2 htG}\left\{2(1+\mu) - \frac{\lambda^2}{4} + \eta^2 + A_0 \operatorname{ch} k_0\eta + B_0 \operatorname{sh} k_0\eta + \right.$$
$$\left. + \sum_{n=1}^{\infty} (A_n \operatorname{ch} k_n\eta + B_n \operatorname{ch} \bar{k}_n\eta + C_n \operatorname{sh} k_n\eta + D_n \operatorname{sh} \bar{k}_n\eta)\right\}. \qquad (5.120)$$

where

$$\left.\begin{aligned} k_0 &= 0.97329, \\ k_1 &= 2.58586 + 1.35990\, i, \\ k_2 &= 5.92300 + 2.15590\, i. \end{aligned}\right\} \qquad (5.121)$$

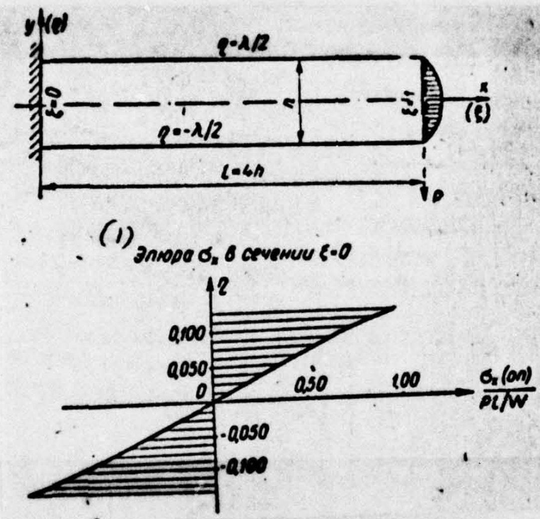

Fig. 26.

Key: (1). Diagram σ_x in section $\xi = 0$.

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Is expressed all the displacement/movements and stresses through the resolving function. For this let us introduce (5.117) in (5.114). This will bring to

$$\left.\begin{aligned}u(\xi,\eta)&=\frac{l}{16}\psi_u(\xi,\beta)\varphi(\eta),\\ v(\xi,\eta)&=\frac{l}{16}\psi_v(\xi,\beta)\varphi(\eta),\\ \sigma_x(\xi,\eta)&=\frac{G}{8}\psi_\sigma(\xi,\beta)\varphi(\eta),\\ \sigma_y(\xi,\eta)&=\frac{G}{8}\psi_y(\xi,\beta)\varphi(\eta),\\ \tau_{xy}(\xi,\eta)&=\frac{G}{8}\psi_\tau(\xi,\beta)\varphi(\eta).\end{aligned}\right\} \quad (5.122)$$

The entering here operator-functions with μ = 0.3 take the form:

$$\begin{aligned}
\psi_u(\xi,\beta) &= (1{,}89-3{,}38\,\beta^2\xi)\,\frac{\cos(1-\xi)\beta}{\beta}-1{,}89\,\frac{\cos(1+\xi)\beta}{\beta}+\\
&\quad+(-3{,}51+1{,}69\xi)\sin(1-\xi)\beta+3{,}51\,(1-\xi)\sin(1+\xi)\beta,\\
\psi_v(\xi,\beta) &= (3{,}51+1{,}69\,\xi)\cos(1-\xi)\beta-3{,}51\,(1-\xi)\cos(1+\xi)\beta+\\
&\quad+(-5{,}4+3{,}38\,\beta^2\xi)\,\frac{\sin(1-\xi)\beta}{\beta}+5{,}4\,\frac{\sin(1+\xi)\beta}{\beta},\\
\psi_\sigma(\xi,\beta) &= 1{,}69\,\beta(1-\xi)\cos(1-\xi)\beta+3{,}51\,\beta(1-\xi)\cos(1+\\
&\quad+\xi)\beta+(2{,}8-3{,}38\,\beta^2\xi)\sin(1-\xi)\beta,\\
\psi_s(\xi,\beta) &= 1{,}69\,(3+\xi)\beta\cos(1-\xi)\beta-3{,}51\,(1-\xi)\beta\cos(1+\\
&\quad+\xi)\beta+(-6.18+3.38\,\beta^2\xi)\sin(1-\xi)\beta+7.02\sin(1+\xi)\beta,\\
\psi_\tau(\xi,\beta) &= (4{,}49-3{,}38\,\beta^2\xi)\cos(1-\xi)\beta+3{,}51\cos(1+\xi)\beta+\\
&\quad+1{,}69\,\beta(1+\xi)\sin(1-\xi)\beta+3{,}51\sin(1+\xi)\beta.
\end{aligned} \qquad (5.123)$$

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Realizing these operators above the function $\varphi(\eta)$ (see (5.122)) according to the rules, presented in chapter 1, let us arrive at the following expressions:

$$U(\xi, \eta) = \frac{6Pl}{\lambda^2 hlE} \Big\{ 2(1-\mu)\xi\eta - \Big(1+\mu)\xi^2\eta + \frac{1}{8}\psi_u(\xi, k_0)(A_0 \operatorname{sh} k_0\eta + B_0 \operatorname{ch} k_0\eta) + \frac{1}{8}\sum_{n=1}^{\infty} [\psi_u(\xi, k_n)(A_n \operatorname{sh} k_n\eta + C_n \operatorname{ch} k_n\eta) + \psi_u(\xi, \bar{k}_n)(B_n \operatorname{sh} \bar{k}_n\eta + D_n \operatorname{ch} \bar{k}_n\eta)] \Big\} \quad (5.124)$$

and so forth.

Here we by formula $G = E/(2(1+\mu))$ pass of modulus of shear G toward the modulus of elasticity E.

Substituting these expressions under boundary conditions (5.112), we obtain the equations

$$A_0\psi_y(\xi, k_0) \operatorname{sh} \frac{k_0\lambda}{2} + \operatorname{Re} \left\{ \sum_{n=1}^{\infty} (a_n + ib_n)\psi_y(\xi, k_n) \operatorname{sh} \frac{k_n\lambda}{2} \right\} = -8[\mu + (2+\mu)\xi]\lambda, \quad (5.125)$$

$$A_0\psi_\tau(\xi, k_0) \operatorname{ch} \frac{k_0\lambda'}{2} + \operatorname{Re} \left\{ \sum_{n=1}^{\infty} (a_n + ib_n)\psi_\tau(\xi, k_n) \operatorname{ch} \frac{k_n\lambda}{2} \right\} = -16 - 24\mu + 16\mu\xi + 8(2+\mu)\xi^2. \quad (5.126)$$

$\eta = \frac{\lambda}{2}$

Let us note that on the strength of the symmetry of conditions with
and $\eta = \frac{-\lambda}{2}$, the coefficients B_n and D_n becomes zero. Therefore will be simplified expressions (5.127) for displacement/movements and stresses. For example, it will be

$$\sigma_x(\xi, \eta) = \frac{6P}{\lambda^2 ht}\Big\{2(1-\xi)\eta + \frac{A_0}{8}\psi_\sigma(\xi, k_0)\,\mathrm{sh}\,k_0\eta + \\ + \frac{1}{8}\mathrm{Re}\left[\sum_{n=1}^{\infty}(a_n + ib_n)\psi_\sigma(\xi, k_n)\,\mathrm{sh}\,k_n\eta\right]\Big\}. \quad (5.127)$$

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Here as in the preceding/previous equations, it is marked

$$A_n = a_n + ib_n. \quad (5.128)$$

Let us take $\lambda = 1/4$ and write out the auxiliary values

$$\left.\begin{aligned} \psi_\sigma(0, k_0) &= +5.162, \\ \psi_\sigma(0, k_1) &= -13.861 - 29.709i, \\ \psi_\sigma(0, k_2) &= +73.771 + 62.202i. \end{aligned}\right\} \quad (5.129)$$

Table 12

η	sh $k_0\eta$	ch $k_0\eta$
0	0	1
0,025	0,02433	1,00030
0,050	0,04868	1,00118
0,075	0,07306	1,00267
0,100	0,09748	1,00474
0,125	0,12196	1,00741

Table 13.

η	sh $k_1\eta$	ch $k_1\eta$
0	0	1
0,025	0,06435 + 0,03406 i	1,00151 + 0,00220 i
0,050	0,12935 + 0,06831 i	1,00604 + 0,00881 i
0,075	0,19114 + 0,10373 i	1,01357 + 0,01987 i
0,100	0,25906 + 0,14013 i	1,02408 + 0,03545 i
0,125	0,32415 + 0,17809 i	1,03752 + 0,05564 i

Table 14.

η	sh $\varkappa_2\eta$	ch $\varkappa_2\eta$
0	0	1
0,025	0,14839 + 0,05446 i	1,00951 + 0,00801 i
0,050	0,29375 + 0,11233 i	1,03810 + 0,03233 i
0,073	0,45298 + 0,17714 i	1,08596 + 0,07389 i
0,100	0,61301 + 0,25255 i	1,15327 + 0,13424 i
0,125	0,78066 + 0,34260 i	1,24036 + 0.21562 i

Equations (5.125) and (5.126) do not make it possible to find precise the value of coefficients. Therefore let us use the same approximate method, as in previous paragraphs: is decomposed the entering these equations functions according to degrees ξ and let us equate zero absolute terms and the coefficients with ξ in both equations, and also the coefficient with ξ^2 in equation (5.126). Thus, let us arrive at the following system:

$$\left.\begin{aligned}
&1{,}08828A_0+1{,}50973\,a_1+2{,}59568\,b_1-9{,}04582\,a_2-\\
&\quad-15{,}3422\,b_2=-2{,}90,\\
&0{,}66461\,A_0-1{,}11473\,a_1+0{,}88025\,b_1+2{,}78427\,a_2-\\
&\quad-1{,}49528\,b_2=-1{,}83333,\\
&5{,}34312\,A_0-8{,}88934\,a_1+7{,}54332\,b_1+24{,}27197\,a_2-\\
&\quad-16{,}6220\,b_2=-14{,}6667,\\
&1{,}91610\,A_0-6{,}93998\,a_1+1{,}71209\,b_1-8{,}15918\,a_2+\\
&\quad+0{,}29678\,b_2=-5{,}4,\\
&0{,}99300\,A_0-4{,}880820\,a_1+0{,}50744\,b_1-19{,}0769\,a_2+\\
&\quad+0{,}37549\,b_2=-2{,}85335.
\end{aligned}\right\}\quad(5.130)$$

Solving it, we find:

$$\left.\begin{aligned}
&A_0=-2{,}75817;\ a_1=-0.00348;\ b_1=-0.05341;\\
&a_2=+0.00518;\ b_2=-0{,}01906.
\end{aligned}\right\}\quad(5.131)$$

Now, after assuming in (5.127) $\xi = 0$, outside in (5.127) the obtained values of coefficients and after using auxiliary values (5.129) and data by Table 12-14, we can compute a series of values in the pinched section (Table 15), where, as usual, it is marked

$$\sigma_{max} = \frac{Pl}{W}. \quad (5.132)$$

According to these data on Fig. 26 is constructed diagram/curve σ in stopping up. This diagram/curve and Table 15, shows that i during precise satisfaction of the conditions of jamming the distribution of normal stresses according to the pinched section remains virtually such, which is accepted in strength of materials.

Table 15.

η	$\sigma_x(0, \eta)/\sigma_{max}$
0	0
0,025	0,176
0,050	0,356
0,075	0,547
0,100	0,753
0,125	0,980

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CONCLUSION.

After feeding result to entire presented above, it is possible to note the following basic results, obtained in work.

1. Is carried out the detailed analysis of the basic works of A. I. Lur'yes and V. Z. Vlasov, whom they expressed and developed the

ideas of the method of the initial functions.

2. Are developed the bases of the theory of the operators in connection with the method of the initial functions:

a) is shown the possibility of double approach to the introduction of the concept of the operators;

b) are based the algebraic, differential and integral actions above the regular, singular and mixed operators; c) is demonstrated the equivalency of the representation of the operators in the closed form and in the form of series, and also the legitimacy of the action above operational series;

d) is obtained a series of the formulas, which express the properties of the operators;

e) are establish/installed specific rules for the actions with the operators, the connected with noninterchangeability mixed operators;

f) are developed the methods of the realization of the operators;

g) is establish/installed that values of the function-operators compulsorily satisfy some partial differential equations.

3. Obtained considerable quantity of formulas, which facilitate the practical application/use of the operators.

4. Is shown the possibility of obtaining some new identities for Bessel functions.

5. Is systematized the overall diagram of the application/use of a method of the initial functions to the solution to two-dimensional boundary-value problems; in this case is assumed the wide use of the operators.

6. Is discovered the equivalency of operational equations and functional equations of the type integrodifferential, differential-difference and so forth.

7. Is obtained the new method of the determination of the particular solutions to nonhomogeneous differential equations in partial derivatives.

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8. Are constructed general solutions for the series of problems of the applied theory of elasticity and strength of materials (two-dimensional problem, the bending of plates, etc.).

9. For facilitating the solution to the specific problems

a) are obtained the resolving equations for the basic cases of support in two-dimensional problem of elasticity theory and with the curvature of rectangular plate;

b) calculated considerable quantity of roots of some frequently being encountered transcendental characteristic equations.

10. For the purpose of the explanation of the application/use of a method of the initial functions some problems (free twisting the rod of semicircular section, the curvature of the pinched on outline/contour plate, the curvature of short cantilever plate, the curvature of long cantilever beam) carried to numerical result.

The method of the initial functions can receive very widespread propagation in theory of elasticity, theoretical aerodynamics and other sciences under the condition of the further development of several the basic for this method questions. The principal directions of this development, apparently, must be the following:

1) obtaining numerical results directly from the operational form of solution, passing transition to usual functional form;

2) the development of the methods of the direct/straight programming of the actions above operational expressions in the high-speed digital computers;

3) obtaining the so-called exact solutions for complex boundary conditions; in this case, apparently, necessary to adhere to the first and third treatments of the resolving equations (page 131).

Author made work in the last/latter direction. The obtained results make it possible to hope that in this direction it is possible to achieve the determined successes.

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APPLICATION/APPENDIX.

Base properties and formulas for the operators.

1. $L(\beta) = L^{+}(\beta) = a_0 + a_1\beta + a_2\beta^2 + \ldots$ - regular operator.

2. $L(\beta) = L^{-}(\beta) = \frac{a_{-1}}{\beta} + \frac{a_{-2}}{\beta^2} + \frac{a_{-3}}{\beta^3} + \ldots$ - singular operator.

3. $L(\beta) = \cdots + \frac{a_{-1}}{\beta} + a_0 + a_1\beta + a_2\beta^2 + \cdots$ - the mixed operator.

4. $L^{+}(\beta)\varphi(\eta) = a_0\varphi + a_1\varphi' + a_2\varphi'' + \cdots$

5. $L^{-}(\beta)\varphi(\eta) = \int_0^{\eta} \varphi(\zeta) \sum_{s=1}^{\infty} \frac{a_{-s}(\eta-\zeta)^{s-1}}{(s-1)!} d\zeta.$

6. $L(\xi, \beta)\varphi(\eta) = \sum_{s=-\infty}^{\infty} a_s(\xi)\beta^s\varphi(\eta)$ - operator-function.

7. $L(\xi, \beta)\{\varphi(\eta)\} = \Phi(\xi, \eta)$

for all those $\varphi(\eta)$, for which makes sense the concrete/specific/actual formula of realization.

8. $L(\xi,\beta+k)\varphi(\eta)=e^{-k\eta}L(\xi,\beta)\{e^{k\eta}\varphi(\eta)\}$.

where k is random.

9. $\dfrac{\partial^n L(\xi,\beta)}{\partial\beta^n}\varphi(\eta)=\sum_{s=0}^{n}(-1)^s C_n^s \eta^s L(\xi,\beta)\{\eta^{n-s}\varphi(\eta)\}$.

Here n ≥ is whole.

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10. If is satisfied condition $L(\xi,\beta)\{\varphi(\eta)\}=\Phi(\xi,\eta)$, then

$$L(\xi,m\beta)\varphi\left(\frac{\eta}{m}\right)=\Phi\left(\xi,\frac{\eta}{m}\right).$$

$m \neq 0$ - any (complex) number.

11. For

$$L_1L_2=L_2L_1$$

it is necessary and sufficiently in order that would be

a) both operators were regular.

b) both operators were singular,

c) $L_1 \cdot L_2 = a_0 + \frac{a_{-1}}{\beta} + \frac{a_{-2}}{\beta} + \dots,$

d) $L_2 = \lambda L_1 + C$.

12. If L_1 (z) L_2 (z) = L_3 (z), then in order that would be L_1 (β) L_2 (β) = L_3 (β), it is necessary and sufficient in order to

a) $L_1 = L_1^+$.

b) $L_2 = a_0 + \frac{a_{-1}}{\beta} + \frac{a_{-2}}{\beta^2} + \dots$.

For example,

$$\sqrt{\beta^2 - k^2} \cdot \frac{1}{\beta^2 - k^2} = \frac{1}{\sqrt{\beta^2 - k^2}},$$

$$\operatorname{sh} \frac{\xi}{\beta} \operatorname{ch} \frac{\xi}{\beta} = \operatorname{ch} \frac{\xi}{\beta} \operatorname{sh} \frac{\xi}{\beta} = \frac{1}{2} \operatorname{sh} \frac{2\xi}{\beta},$$

$$\frac{1}{\beta - 1} (\beta^2 - 1) \neq \beta + 1,$$

$$e^{\frac{\xi}{\beta}} \cdot e^{\xi\alpha} \neq e^{\xi\left(\frac{1}{\beta} + \alpha\right)}.$$

13. If in the interval/gap of change ξ function $a_s(\xi)$ are evenly continuous, then

$$\lim_{\xi\to\xi_1} L(\xi,\beta)\varphi(\eta) = L(\xi_1,\beta)\varphi(\eta),$$

$$\int L(\xi,\beta)\varphi(\eta)\,d\xi = \left(\int L(\xi,\beta)\,d\xi\right)\{\varphi(\eta)\},$$

$$\frac{\partial^m}{\partial\xi^m}[L(\xi,\beta)\varphi(\eta)] = \frac{\partial^m L}{\partial\xi^m}\varphi.$$

14.

$$L^+(\beta)e^{\beta m} = L^+(m)e^{\beta m} \quad m - {}^{(1)}\text{любое},$$

$$L^-(\beta)e^{\beta m} + L^-(m)e^{\beta m}.$$

Key: (1). any.

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15. $$L^{+}(\xi, \beta+k)\{\eta^{n}\} = \left(\eta+\frac{\partial}{\partial k}\right)^{n} L(\xi, k).$$

16. For the even operators

$$L(\xi, \beta)\left\{\genfrac{}{}{0pt}{}{\mathrm{ch}}{\mathrm{sh}} m\eta\right\} = L(\xi, m) \genfrac{}{}{0pt}{}{\mathrm{ch}}{\mathrm{sh}} m\eta,$$

$$L(\xi, \beta)\left\{\genfrac{}{}{0pt}{}{\cos}{\sin} m\eta\right\} = L(\xi, im) \genfrac{}{}{0pt}{}{\cos}{\sin} m\eta,$$

$$L(\xi, \beta)\left\{\eta \genfrac{}{}{0pt}{}{\mathrm{ch}}{\mathrm{sh}} m\eta\right\} = L(\xi, m)\, \eta \genfrac{}{}{0pt}{}{\mathrm{ch}}{\mathrm{sh}} m\eta + \frac{\partial L(\xi, m)}{\partial m} \genfrac{}{}{0pt}{}{\mathrm{sh}}{\mathrm{ch}} m\eta;$$

for the odd operators

$$L(\xi, \beta)\left\{\genfrac{}{}{0pt}{}{\mathrm{ch}}{\mathrm{sh}} m\eta\right\} = L(\xi, m) \genfrac{}{}{0pt}{}{\mathrm{sh}}{\mathrm{ch}} m\eta,$$

$$L(\xi, \beta)\left\{\genfrac{}{}{0pt}{}{\cos}{\sin} m\eta\right\} = \pm iL(\xi, im) \genfrac{}{}{0pt}{}{\sin}{\cos} m\eta,$$

$$L(\xi, \beta)\left\{\eta \genfrac{}{}{0pt}{}{\mathrm{ch}}{\mathrm{sh}} m\eta\right\} = L(\xi, m)\, \eta \genfrac{}{}{0pt}{}{\mathrm{sh}}{\mathrm{ch}} m\eta + \frac{\partial L(\xi, m)}{\partial m} \genfrac{}{}{0pt}{}{\mathrm{ch}}{\mathrm{sh}} m\eta.$$

for example,

$$\cos\beta\,\{\eta\,\mathrm{ch}\, m\eta\} = \cos m \cdot \eta\,\mathrm{ch}\, m\eta - \sin m \cdot \mathrm{sh}\, m\eta.$$

17.

$$e^{\xi\beta}\varphi(\eta) = \varphi(\eta+\xi).$$

$$\sin\xi\beta\varphi(\eta) = \operatorname{Im}\{\varphi(\eta+i\xi)\},$$

$$\cos\xi\beta\varphi(\eta) = \operatorname{Re}\{\varphi(\eta+i\xi)\},$$

$$\frac{\sin\xi\beta}{\beta}\varphi(\eta) = \frac{1}{2i}\int_{\eta-i\xi}^{\eta+i\xi} e^{k(\xi-\eta)}\varphi(\zeta)\,d\zeta.$$

$$\frac{\xi\beta\,\mathrm{ch}\,\xi\beta - \mathrm{sh}\,\xi\beta}{\beta^3}\varphi(\eta) = \frac{1}{2}\int_{\eta-\xi}^{\eta+\xi}(\zeta-\eta)\varphi(\zeta)\,d\zeta.$$

18.

$$\frac{1}{(\beta-k)^n}\varphi(\eta) = \frac{e^{k\eta}}{(n-1)!}\int_0^\eta e^{-k\zeta}(\eta-\zeta)^{n-1}\varphi(\zeta)\,d\zeta.$$

$$\frac{1}{\beta-k}\{1\} = \frac{1}{k}(e^{k\eta}-1), \qquad \frac{1}{\beta-k}e^{m\eta} = \frac{e^{m\eta}-e^{k\eta}}{m-k}$$

$$\frac{1}{\beta^2-k^2}\varphi(\eta) = \frac{1}{k}\int_0^\eta \mathrm{sh}\,k(\eta-\zeta)\,\varphi(\zeta)\,d\zeta.$$

$$\frac{1}{\beta^4-k^4}\varphi(\eta) = \frac{1}{2k^3}\int_0^\eta [\mathrm{sh}\,k(\eta-\zeta)-\sin k(\eta-\zeta)]\,\varphi(\zeta)\,d\zeta.$$

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19.

$$\ln\left(1-\frac{1}{\beta}\right)e^{m\eta} = e^{m\eta}\left\{\ln\left(1-\frac{1}{m}\right) - Ei(-m\eta) + Ei[-(m-1)\eta]\right\}.$$

$$\frac{1}{\sqrt{\beta^2+k^2}}\varphi(\eta) = \int_0^\eta \varphi(\zeta)\, I_0[k(\eta-\zeta)]\, d\zeta.$$

20.

$$\sqrt{\beta^2+k^2}\,\varphi(\eta) = \varphi'(\eta) + k\int_0^\eta \frac{\varphi(\zeta)}{\eta-\zeta} J[k(\eta-\zeta)]\, d\zeta.$$

$$e^{k/\beta}\varphi(\eta) = \varphi(0)\, I_0(2\sqrt{k\eta}) + \int_0^\eta I_0(2\sqrt{k(\eta-\zeta)})\,\varphi'(\zeta)\, d\zeta.$$

$$e^{-\xi\sqrt{\beta^2+k^2}}\varphi(\eta) = \varphi(\eta-\xi) + k\xi\int_{\eta-\xi}^{0} \frac{J\left[\frac{k}{2}\sqrt{(\eta-\zeta)^2-\xi^2}\right]}{\sqrt{(\eta-\zeta)^2-\xi^2}}\varphi(\zeta)\, d\zeta.$$

$$\cos\xi\sqrt{\beta^2+k^2}\,\varphi = \frac{\varphi(\eta+i\xi)+\varphi(\eta-i\xi)}{2} +$$

$$+ ik\xi\int_{\eta-i\xi}^{\eta+i\xi} \frac{J\left[\frac{k}{2}\sqrt{(\eta-\zeta)^2+\xi^2}\right]}{\sqrt{(\eta-\zeta)^2+\xi^2}}\varphi(\zeta)\, d\zeta.$$

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